Algebra is Half the Battle: Verifying Presentations for Graded Unipotent Chevalley Groups

ITP 2025

Eric Wang

Arohee Bhoja

Cayden Codel

Noah Singer

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Work done at

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Undergraduates
(Wrote the proofs)

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(Knew the maths)

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- Automation in need of automation `lin_id_inv_thms`

Warning!

I know about the math we formalized, but not much about the surrounding research context

What we proved

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We proved in Lean that three specific groups can be defined using a canonically smaller set of equations than previously known.

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More specifically, we showed that the A_3, B_3-small, and B_3-large graded unipotent Chevalley groups presented by the “weak” Steinberg relations are isomorphic to the groups presented by the “full” Steinberg relations.

Our proof strategy was to show that each full relation could be derived from the weak relations. We derived each relation by solving one or more group rewriting problems.

Why we proved it

Why we proved it

Cornell University

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Mathematics > Group Theory

[Submitted on 8 Nov 2024]

Coboundary expansion inside Chevalley coset complex HDXs

Ryan O'Donnell, Noah G. Singer

Recent major results in property testing~\cite{BLM24,DDL24} and PCPs~\cite{BMV24} were unlocked by moving to high-dimensional expanders (HDXs) constructed from $\widetilde{C}_d$-type buildings, rather than the long-known $\widetilde{A}_d$-type ones. At the same time, these building quotient HDXs are not as easy to understand as the more elementary (and more symmetric/explicit) \emph{coset complex} HDXs constructed by Kaufman--Oppenheim~\cite{KO18} (of A_d-type) and O'Donnell--Pratt~\cite{OP22} (of B_d-, C_d-, D_d-type). Motivated by these considerations, we study the B_3-type generalization of a recent work of Kaufman--Oppenheim~\cite{KO21}, which showed that the A_3-type coset complex HDXs have good 1-coboundary expansion in their links, and thus yield 2-dimensional topological expanders.
The crux of Kaufman--Oppenheim's proof of 1-coboundary expansion was: (1)~identifying a group-theoretic result by Biss and Dasgupta~\cite{BD01} on small presentations for the A_3-unipotent group over~$\mathbb{F}_q$; (2)~``lifting'' it to an analogous result for an A_3-unipotent group over polynomial extensions~$\mathbb{F}_q[x]$.
For our B_3-type generalization, the analogue of~(1) appears to not hold. We manage to circumvent this with a significantly more involved strategy: (1)~getting a computer-assisted proof of vanishing 1-cohomology of B_3-type unipotent groups over~$\mathbb{F}_5$; (2)~developing significant new ``lifting'' technology to deduce the required quantitative 1-cohomology results in B_3-type unipotent groups over $\mathbb{F}_{5^k}[x]$.

Comments:	130 pages
Subjects:	**Group Theory (math.GR)**; Discrete Mathematics (cs.DM)
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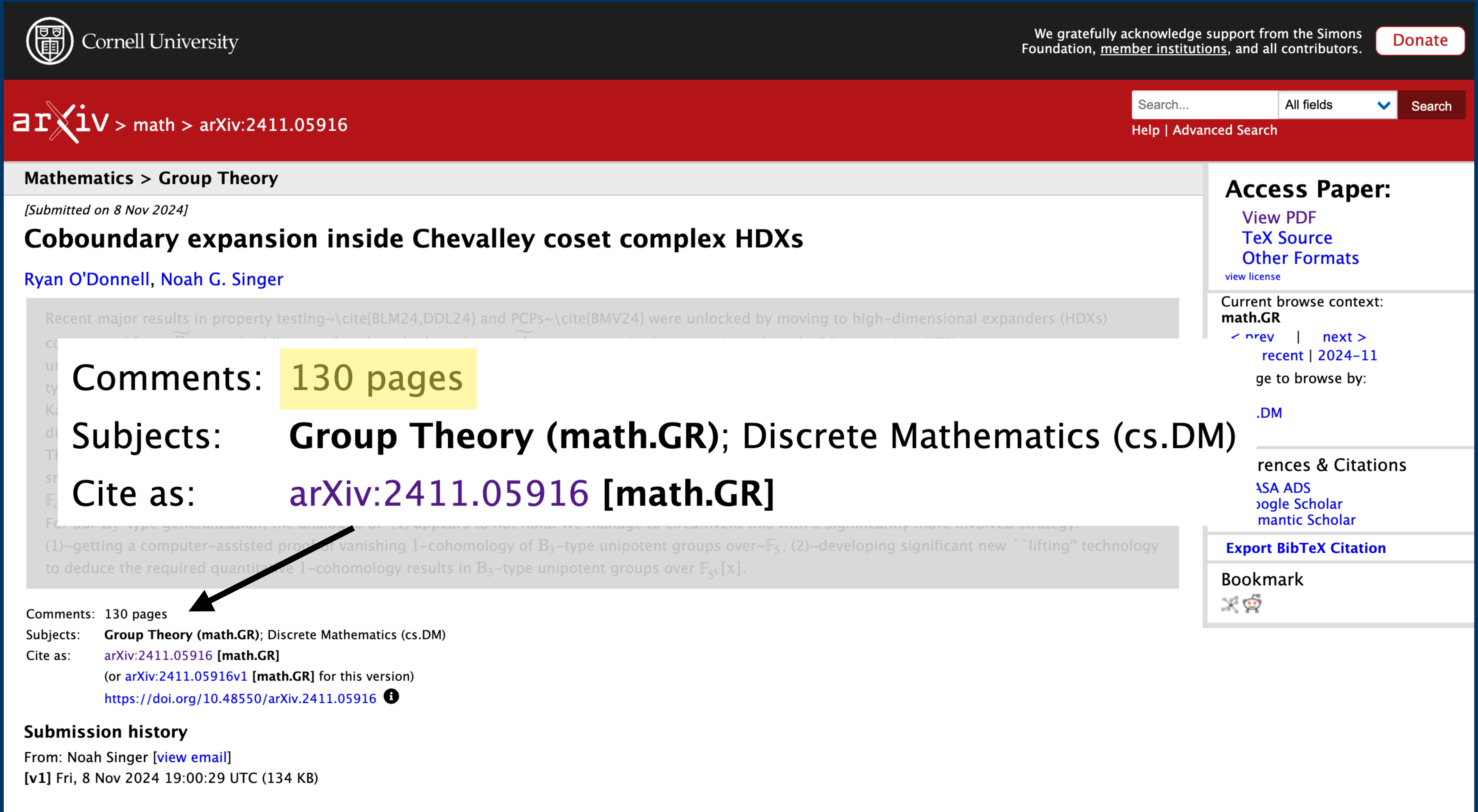
Why we proved it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  - Constants, negative signs, delicate computations

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- To verify a result involving hundreds of calculations
    - Constants, negative signs, delicate computations
    - Noah: “What if I made a mistake somewhere?”
- To lay the groundwork for similar verifications
    - Lots of future work left!

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Higher-dimensional expanders

(Specifically, topological expanders)

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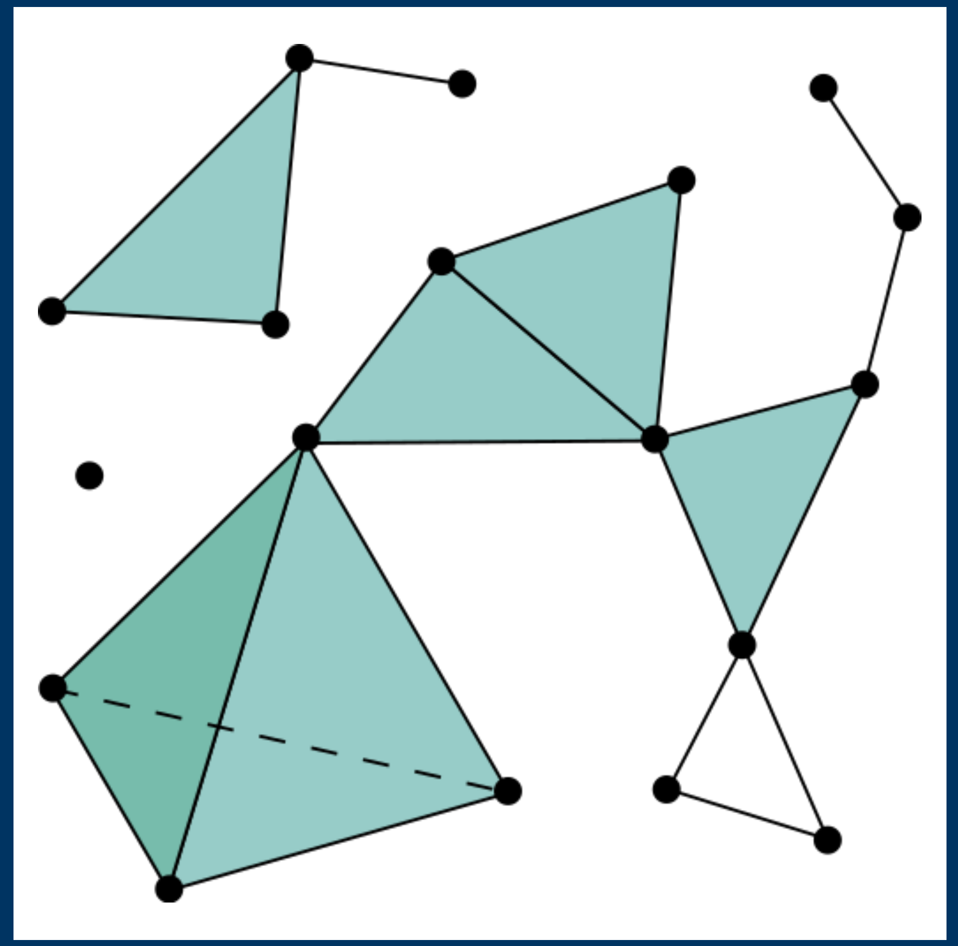
Simplicial complexes



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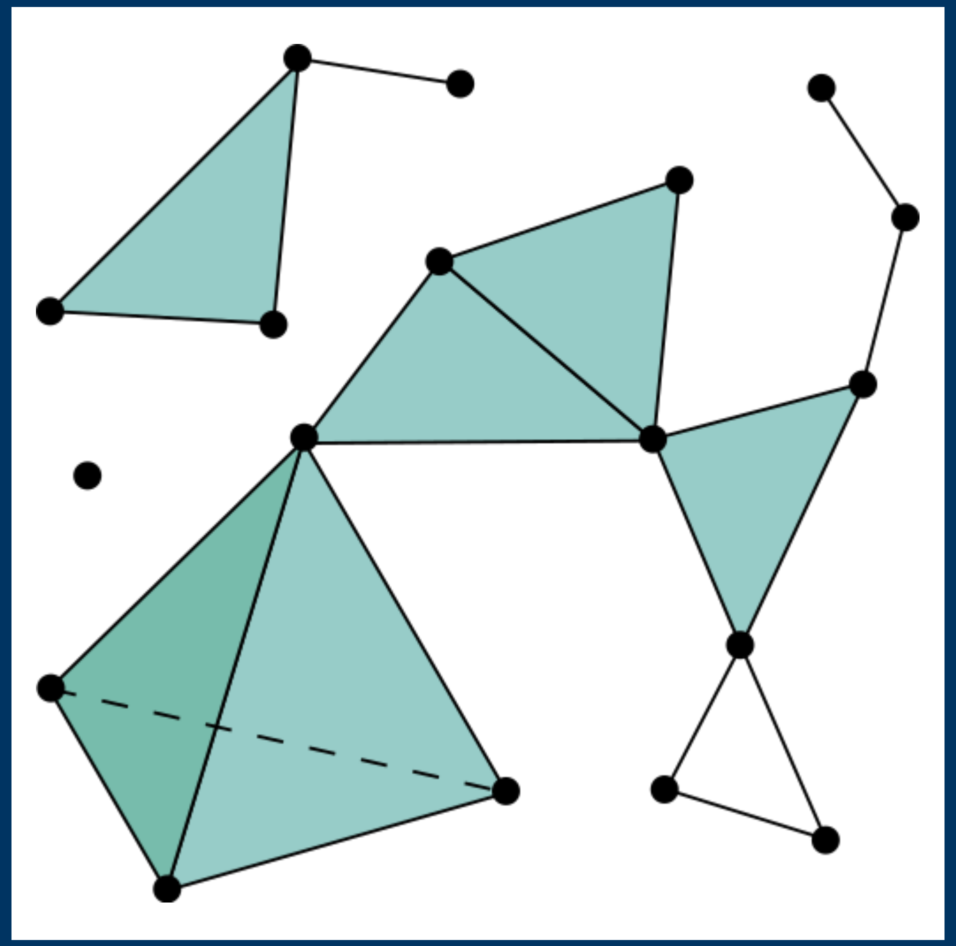

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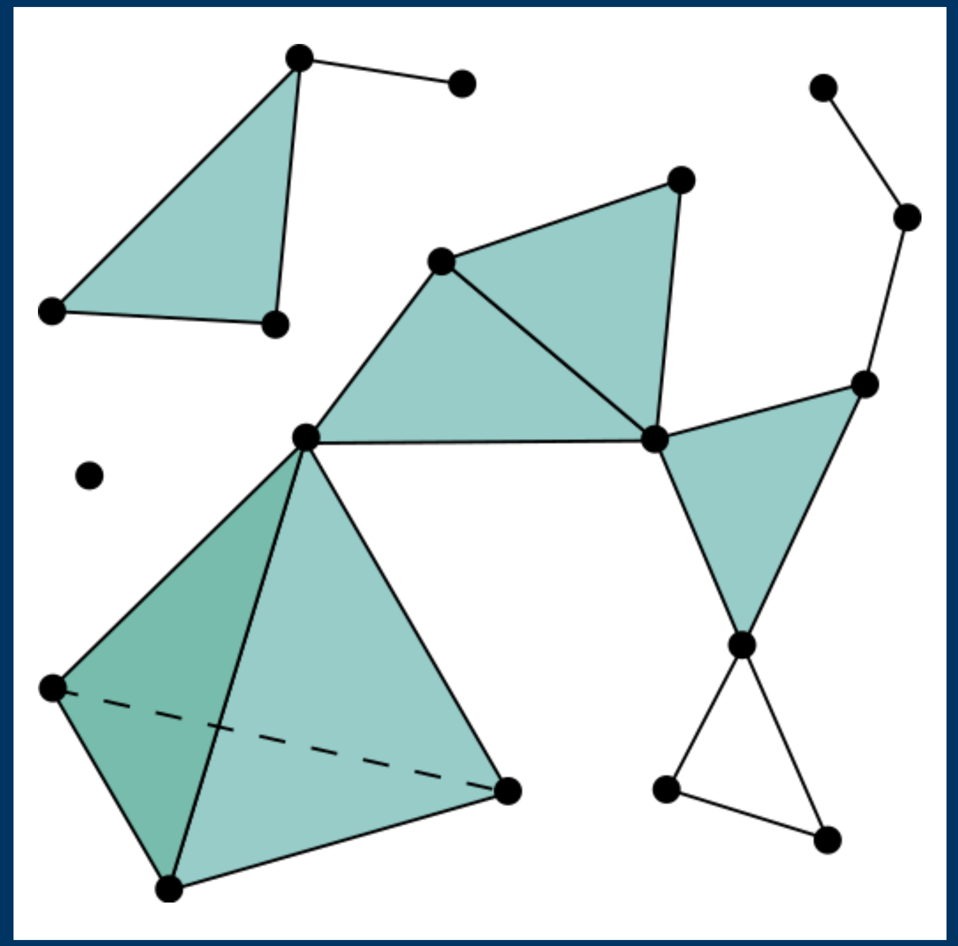

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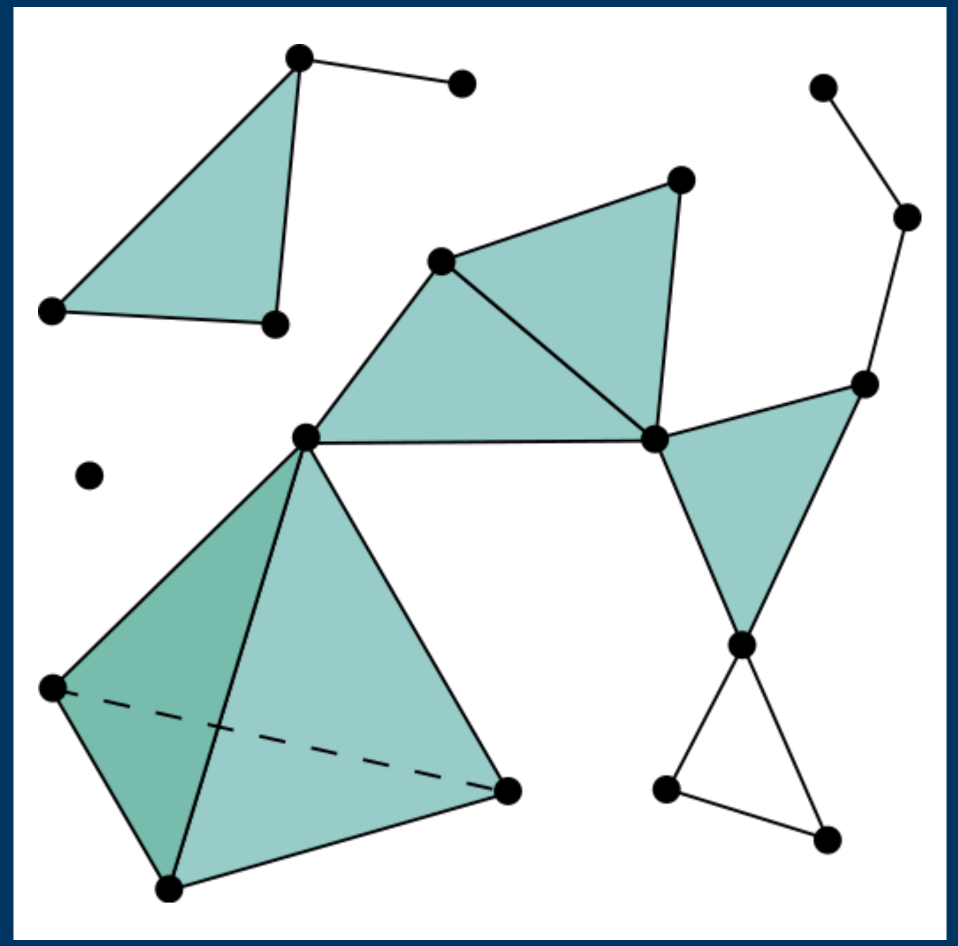

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Chevalley groups
defined by a good set of Steinberg relations

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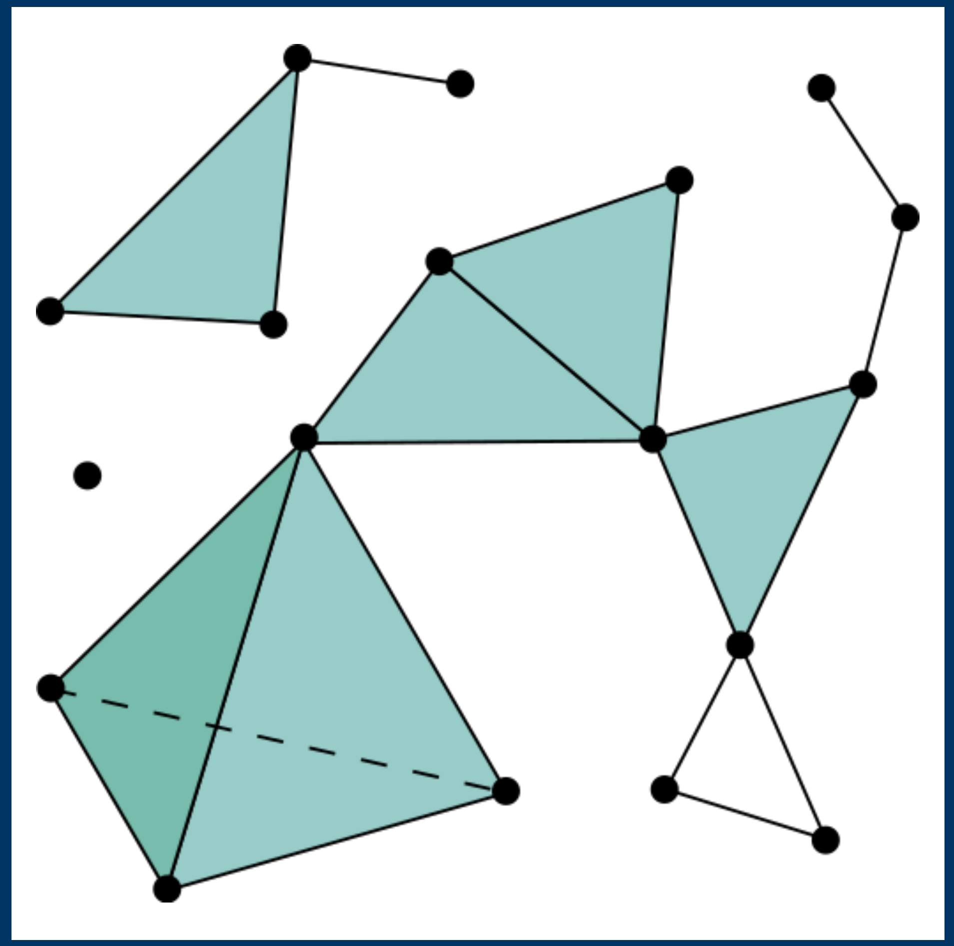

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We formalized this part

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  - Useful for (quantum) error correction, local property testing, and higher-dimensional geometry

Why work on this problem?

- To construct higher-dimensional expanders
  - Useful for (quantum) error correction, local property testing, and higher-dimensional geometry
- To do basic research in group theory

Chevalley groups

Chevalley groups

- Similar to Lie groups

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- Defined on square matrices containing field elements

Chevalley groups

- Similar to Lie groups
- Defined on square matrices containing field elements
- Express nice symmetries and geometric properties

Chevalley groups

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$$\{\zeta, t\} \qquad \{\zeta, i, t\}$$

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Root (vector)

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```
structure GradedChevalleyGenerator
    (Φ : Type TΦ) [PositiveRootSystem Φ]
    (R : Type TR) [Ring R]
where
  ζ : Φ -- root
  i : ℕ -- degree
  hi : i ≤ height ζ
  t : R -- field coefficient
```

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  - Commutator: For any pair of roots ζ and η, the commutator is a product of generators of the following form, for Chevalley constants a, b, and $C^{a,b}_{\zeta,\eta} \in [-3,3]$:

$$[\{\zeta, i, t\}, \{\eta, j, u\}] = \Pi \ \{a\zeta + b\eta, ai + bj, C^{a,b}_{\zeta,\eta} t^a u^b\}$$

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These are (some of the) Steinberg relations

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Equivalently, G is the free group on S modulo the normal closure of R in S

Presentations give a succinct way of specifying groups

An example calculation

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Assume the weak Steinberg relations. Then we want to show the full relation

$$\{\beta + \psi + \omega, i, t\}\{\beta + \psi + \omega, i, u\} = \{\beta + \psi + \omega, i, t + u\}$$

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&= \{\beta, i_1, t\}\{\beta+\omega+\psi, i, u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\} \\
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&= \{\beta, i_1, t+u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -u\}\{\psi+\omega, i_2, 0\}\{\beta, i_1, -t\}\{\psi+\omega, i_2, -1\} \\
&= \{\beta, i_1, t+u\}\{\psi+\omega, i_2, 1\}\{\beta, i_1, -(t+u)\}\{\psi+\omega, i_2, -1\} \\
&= \{\beta+\psi+\omega, i_1+i_2, t+u\}
\end{aligned}$$

An example calculation

Assume the weak Steinberg relations. Then we want to show the full relation

$$\{\beta+\psi+\omega, i, t\}\{\beta+\psi+\omega, i, u\} = \{\beta+\psi+\omega, i, t+u\}$$

Proof. Decompose i arbitrarily into $i = i_1 + i_2$. Then

$$
\begin{aligned}
&\{\beta+\psi+\omega, i, t\}\{\beta+\psi+\omega, i, u\} \\
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$$

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Proof. Decompose i arbitrarily into $i = i_1 + i_2$. Then …

```
@[simp, chev_simps]
theorem lin_of_βψω : ∀ (i : N) (t u : F),
    {βψω, i, t} * {βψω, i, u} = {βψω, i, t + u} := by
  intro i hi t u
  rcases decompose i with ⟨i₁, i₂⟩    -- i = i₁ + i₂
  grw [ expr_βψω_as_β_ψω_β_ψω,                          -- Line 1
        expr_βψω_ψω_as_ψω_βψω, expr_βψω_β_as_β_βψω,     -- Line 2
        expr_βψω_ψω_as_ψω_βψω,                          -- Line 3
        mul_one u, expr_βψω_as_β_ψω_β_ψω,               -- Line 5
        mul_one (t + u), expr_βψω_as_β_ψω_β_ψω ]        -- Line 6
```

An example calculation

Assume the weak Steinberg relations. Then we want to show the full relation

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Proof. Decompose i arbitrarily into $i = i_1 + i_2$. Then …

```
@[simp, chev_simps]
theorem lin_of_βψω : lin_of_root((weakB3SmallGraded F).project, βψω) :=
by
  intro i hi t u
  rcases decompose 1 2 i hi with ⟨i₁, i₂, rfl, hi₁, hi₂⟩
  rw [←mul_one t, expr_βψω_as_β_ψω_β_ψω Fchar hi₁ hi₂]
  grw [←expr_βψω_ψω_as_ψω_βψω Fchar, ←expr_βψω_β_as_β_βψω Fchar,
  ←expr_βψω_ψω_as_ψω_βψω Fchar]
  rw [←mul_one u]
  grw [expr_βψω_as_β_ψω_β_ψω Fchar hi₁ hi₂]
  rw [←mul_one (t + u)]
  grw [expr_βψω_as_β_ψω_β_ψω Fchar hi₁ hi₂]
  ring_nf
```

Our proof strategy

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Almost all of the proofs in our formalization look like that calculation.

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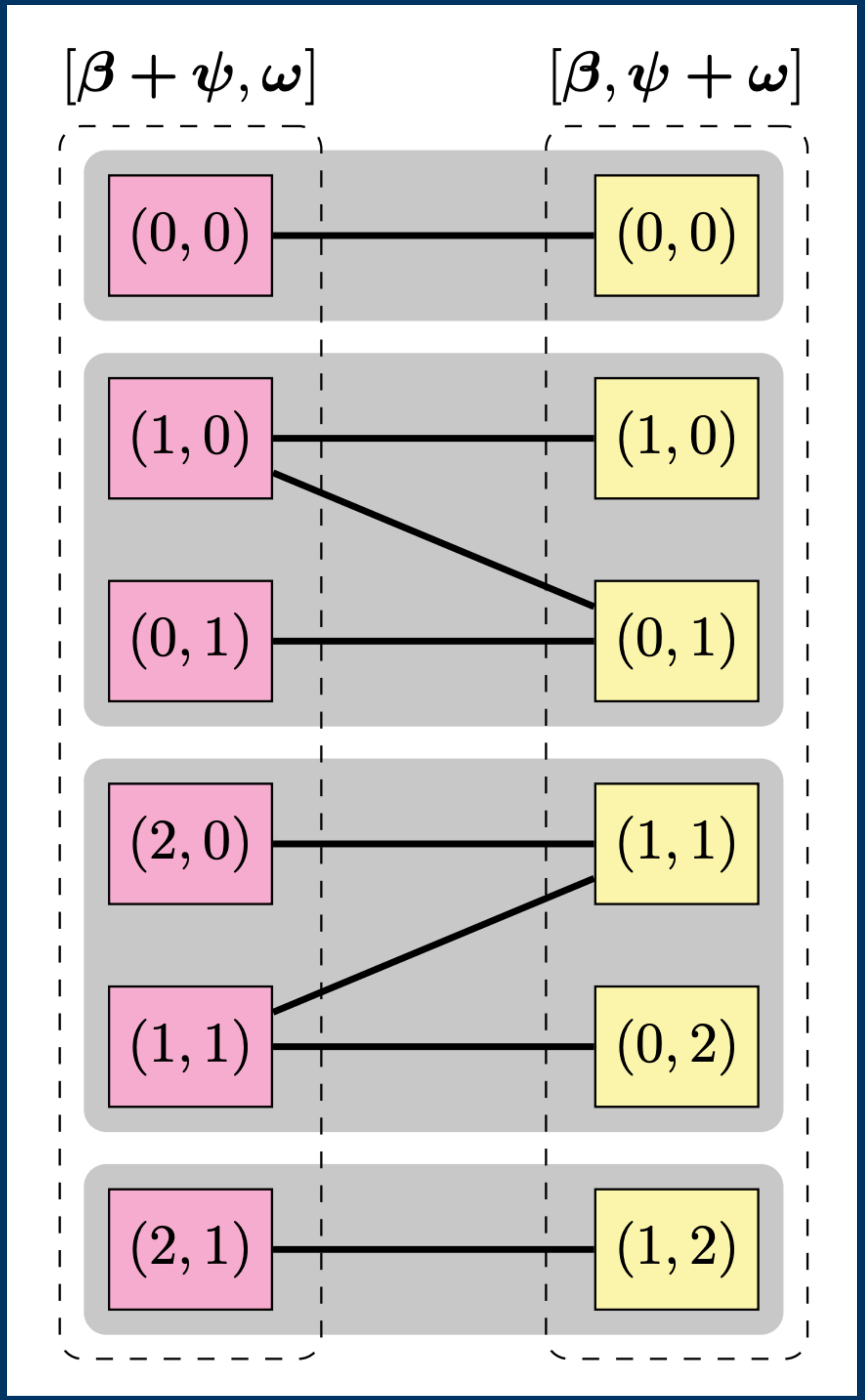

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Sometimes it's worse:

```
theorem expand_βψω_as_commutator_of_βψ_ω :
  forall_ij_tu 2 1, ⸨βψω, i + j, 2 * t * u⸩ = ⁅⸨βψ, i, t⸩, ⸨ω, j, u⸩⁆ := by
  intro i j hi hj t u
  match i, j with
  | 0, 0 => rw [expr_βψω_as_comm_of_βψ_ω_00 Fchar]
  | 0, 1 => rw [expr_βψω_as_comm_of_βψ_ω_01 Fchar]
  | 1, 0 => rw [expr_βψω_as_comm_of_βψ_ω_10 Fchar]
  | 1, 1 => rw [expr_βψω_as_comm_of_βψ_ω_11 Fchar]
  | 2, 0 => rw [expr_βψω_as_comm_of_βψ_ω_20 Fchar]
  | 2, 1 => rw [expr_βψω_as_comm_of_βψ_ω_21 Fchar]
```

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Sometimes it's better (i.e. we get proofs for free):

```
-- height 2 (reflection of height 1)
declare_B3Small_reflected_thm F b3small_valid βψω β ψω const 1 heights 2 1 1 to 1 0 1
declare_B3Small_reflected_thm F b3small_valid βψω β ψω const 1 heights 2 0 2 to 1 1 0
declare_B3Small_reflected_thm F b3small_valid βψω βψ ω const 2 heights 2 2 0 to 1 0 1
declare_B3Small_reflected_thm F b3small_valid βψω βψ ω const 2 heights 2 1 1 to 1 1 0
declare_B3Small_reflected_thm F b3small_valid βψω β ψω const 1 heights 3 1 2 to 0 0 0
declare_B3Small_reflected_thm F b3small_valid βψω βψ ω const 2 heights 3 2 1 to 0 0 0
```

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In this paper, we work with root systems

$$\alpha := e_2 - e_3 \qquad \psi := e_3$$

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inductive B3SmallPosRoot
  | β | ψ | ω | βψ | ψω | β2ψ | βψω
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Some type isomorphisms are better than others

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```
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declare_B3Small_lin_id_inv_thms F ψ
declare_B3Small_lin_id_inv_thms F ω
declare_B3Small_lin_id_inv_thms F βψ
declare_B3Small_lin_id_inv_thms F ψω
declare_B3Small_lin_id_inv_thms F β2ψ

declare_B3Small_trivial_span_of_root_pair_thms F β βψ
declare_B3Small_trivial_span_of_root_pair_thms F β β2ψ
declare_B3Small_trivial_span_of_root_pair_thms F ψ β2ψ
declare_B3Small_trivial_span_of_root_pair_thms F βψ β2ψ
declare_B3Small_trivial_span_of_root_pair_thms F β ω
declare_B3Small_trivial_span_of_root_pair_thms F ψ ψω
declare_B3Small_trivial_span_of_root_pair_thms F ω ψω

declare_B3Small_single_span_of_root_pair_thms F ψ ω ψω 2
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/-! ### Mixed-degree theorem for specific roots -/

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```

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Problem: need to declare the macros for each root system

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Solution: a macro to declare macros?

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$$a * (b * c) = b * (a * c)$$

$$\{\beta, i, t\} \cdot \{\beta, i, -t\} \implies \{\beta, i, 0\} \implies 1$$

Other pain points

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Presentation relations vs. equations

Other pain points

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$$[\{\beta, i, t\}, \{\beta + \psi, j, u\}] = 1$$

$$\Leftrightarrow$$

$$\{\beta, i, t\} \cdot \{\beta + \psi, j, u\} \cdot \{\beta, i, -t\} \cdot \{\beta + \psi, j, -u\} = 1$$

$$\Leftrightarrow$$

$$\{\beta, i, t\} \cdot \{\beta + \psi, j, u\} = \{\beta + \psi, j, u\} \cdot \{\beta, i, t\}$$

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Group/field arithmetic often needed massaging to discharge with automation

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```
have aux₁ : 2 * (u / (2 * t)) = u / t := by ring_nf; field_simp; group
have aux₂ : u / (2 * t) * (u / (2 * t)) = (u * u) / (4 * (t * t)) := by
  ring_nf; simp only [inv_pow, mul_eq_mul_left_iff, inv_inj, mul_eq_zero, ne_eq,
    OfNat.ofNat_ne_zero, not_false_eq_true, pow_eq_zero_iff, inv_eq_zero];
  left
  rw [pow_two, mul_two, two_add_two_eq_four]
```

Thank you for your attention

`https://github.com/singerng/steinberg-formalization`

$\{\zeta,\ t\}$

$\{\zeta,\ i,\ t\}$

Degree

Root (vector)

Field element in $\mathbb{F}_q$

$$\alpha \mid \beta \mid \alpha + \beta$$

theorem lin_of_α

$$a * (b * c) = (a * b) * c$$

lin_id_inv_thms

What we proved

Nonnegative span of a set of
basis vectors

Polynomials with
degrees at most d

Graded unipotent Chevalley groups:
A_3, B_3-small, and B_3-large

We showed through calculation that three specific groups
can be defined by a canonically smaller set of equations
than previously known.

Defined by the presentation on

Steinberg relations