A Natural Language Formalization of Perfectoid Rings in $\mathbb{N}$aproche

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Perfectoid rings as defined in the theory of perfectoid spaces by P. Scholze (*Etále Cohomology of Diamaonds*, arXiv):

Definition *A Tate ring R is* perfectoid *if R is complete, uniform, i.e. $R^o \subset R$ is bounded, and there exists a pseudo-uniformizer $\varpi \in R$ such that $\varpi^p | p$ in R^o and the Frobenius map*

$$\Phi \colon R^o / \varpi \to R^o / \varpi^p \colon x \mapsto x^p$$

is an isomorphism.

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Formalization in the $\mathbb{N}$aproche Natural language proof checking system:

Definition. R *is perfectoid iff* R *is complete and uniform and there exists a pseudouniformizer* ϖ *of* R *such that* $\varpi^{p,R} | p^{[R]}$ *in* R^o *within* R *and*

$$\Phi^R\colon R^o / \varpi \cong R^o / \varpi^{p,R}.$$

$\mathbb{N}$aproche accepts and proof-checks texts in a controlled natural language ForTheL (<u>For</u>mula <u>The</u>ory <u>L</u>anguage). Text are written in LaTeX and can be typeset as mathematical articles.

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within R. Indeed $(x-y)^p - t = x^p - y^p$. □

This map also respects the ring operations modulo ϖ and ϖ^p.

Lemma 286. $\Phi(0) = 0$.

Lemma 287. $\Phi(1) = 1$.

Lemma 288. Let ϖ be an element of R^o such that $(\varpi^p)|p^{[R]}$ in R^o within R. Let x, y be elements of R^o. Then

$\Phi(x+y) \equiv_{R^o} \Phi(x) + \Phi(y) \bmod \varpi^p$ within R.

Proof. $p^{[R]}$ divides $(x+y)^p - (x^p + y^p)$ in R^o within R. Then ϖ^p divides $(x+y)^p - (x^p + y^p)$ in R^o within R.

□

Lemma 289. Let ϖ be an element of R^o such that $(\varpi^p)|p^{[R]}$ in R^o within R. Let x be an element of R^o. Then

$\Phi(-x) \equiv_{R^o} -\Phi(x) \bmod \varpi^p$ within R.

Proof.

$\Phi(x + (-x)) \equiv_{R^o} \Phi(x) + \Phi(-x) \bmod \varpi^p$ within R.

ϖ^p divides $0 - (\Phi(x) + \Phi(-x))$ in R^o within R. [timelimit 10] ϖ^p divides $-\Phi(x) - \Phi(-x)$ in R^o within R. [timelimit 3] □

Lemma 290. Let ϖ be an element of R^o such that $(\varpi^p)|p^{[R]}$ in R^o within R. Let x, y be elements of R^o. Then

$\Phi(x \cdot y) \equiv_{R^o} \Phi(x) \cdot \Phi(y) \bmod \varpi^p$ within R.

Proof. $\Phi(x \cdot y) = (x \cdot y)^p = x^p \cdot y^p = \Phi(x) \cdot \Phi(y)$. □

A perfectoid ring requires the Frobenius map to be an isomorphism So far we have established that it is a homomorphism. To express the crucial isomorphism property one would ordinarily apply a general predicate for ring congruence to the rings R^o/a and R^o/b. To cut things short, we (slightly miss-)use the notation $\Phi : S/a \cong T/b$ with LaTeX source

```
$\Phi^{R} : S / a \cong T / b$
```

by defining its meaning in terms of congruences using the parameters S, a, T, b.

Definition 291. Let $S, T \subseteq R$. Let $a \in S$ and $b \in T$. $\Phi : S/a \cong T/b$ iff (for every $x, y \in S$ if $\Phi(x) \equiv_T \Phi(y) \bmod b$ within R then $x \equiv_S y \bmod a$ within R) and (for every $z \in T$ there exists $w \in S$

such that $z \equiv_T \Phi(w) \bmod b$ within R).

14 Perfectoid rings

Now all ingredients are prepared for defining perfectoid rings in $\mathbb{N}$aproche:

Let R denote a Tate ring.

Lemma 292. Let R be complete and ϖ be a pseudouniformizer of R. Then ϖ, ϖ^p do not divide 1 in R^o within R.

Proof. ϖ does not divide 1 in R^o within R.

Assume that ϖ^p divides 1 in R^o within R. Take $b \in R^o$ such that $\varpi^p \cdot b = 1$. Let $q = p - 1$. Then $\varpi^p = \varpi \cdot \varpi^q$. $\varpi \cdot (\varpi^q \cdot b) = (\varpi \cdot \varpi^q) \cdot b = 1$. [timelimit 6] Then ϖ divides 1 in R^o within R. Indeed $\varpi^q \in R^o$. [timelimit 3] □

In this case the quotients R^o/ϖ and R^o/ϖ^p are well-defined rings, and one can define:

Definition 293. R is perfectoid iff R is complete and uniform and there exists a pseudouniformizer ϖ of R such that $\varpi^p|p^{[R]}$ in R^o within R and

$$\Phi : R^o/\varpi \cong R^o/\varpi^p.$$

The present formalization has mainly been directed towards the definition of perfectoid rings in a readable and proof-checked mathematical language. We do not pursue the theory of perfectoid rings any further and we do not consider examples. If one wanted to do so one would have to refine and considerably expand the previous developments.

References

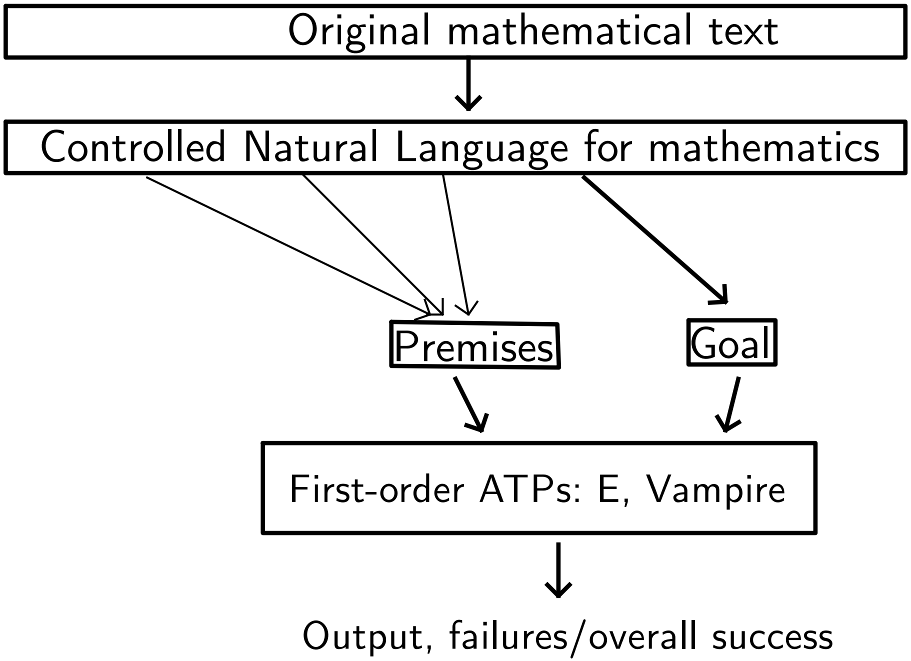
[1] Kevin Buzzard, Johan Commelin, and Patrick Massot: Formalising perfectoid spaces. CPP 2020: 299-312. arXiv:1910.12320. Also: `https://leanprover-community.github.io/lean-perfectoid-spaces/`

[2] Anonymous

[3] R. Huber: Continuous Valuations. Mathematische Zeitschrift, Springer-Verlag, 1993. `http://virtualmath1.stanford.edu/~conrad/Perfseminar/refs/Hubercontval.pdf`

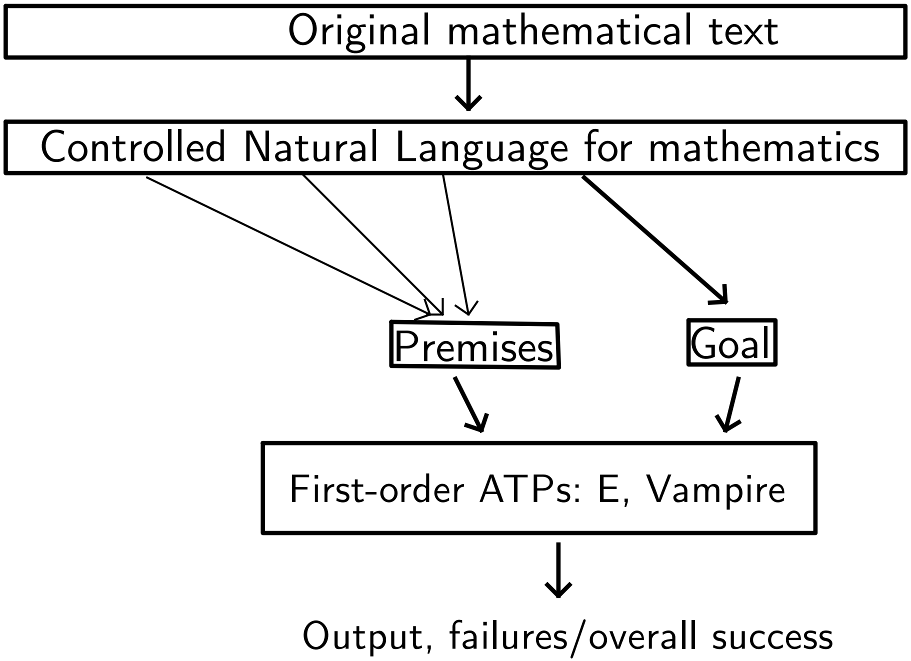
[4] The Isabelle homepage: `https://isabelle.in.tum.de/`

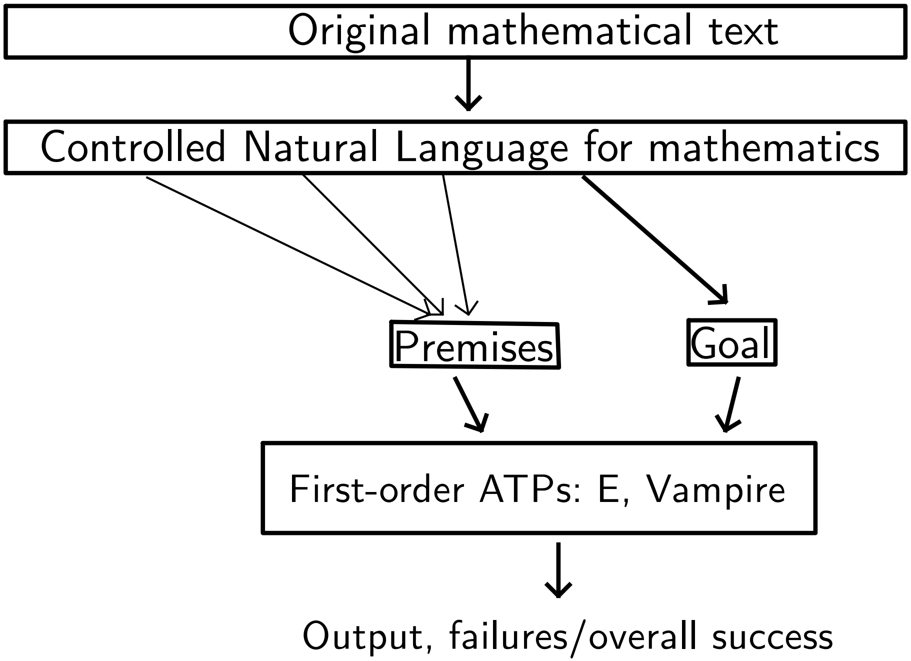
Naproche



Original mathematical text
Controlled Natural Language for mathematics
Premises
Goal
First-order ATPs: E, Vampire
Output, failures/overall success

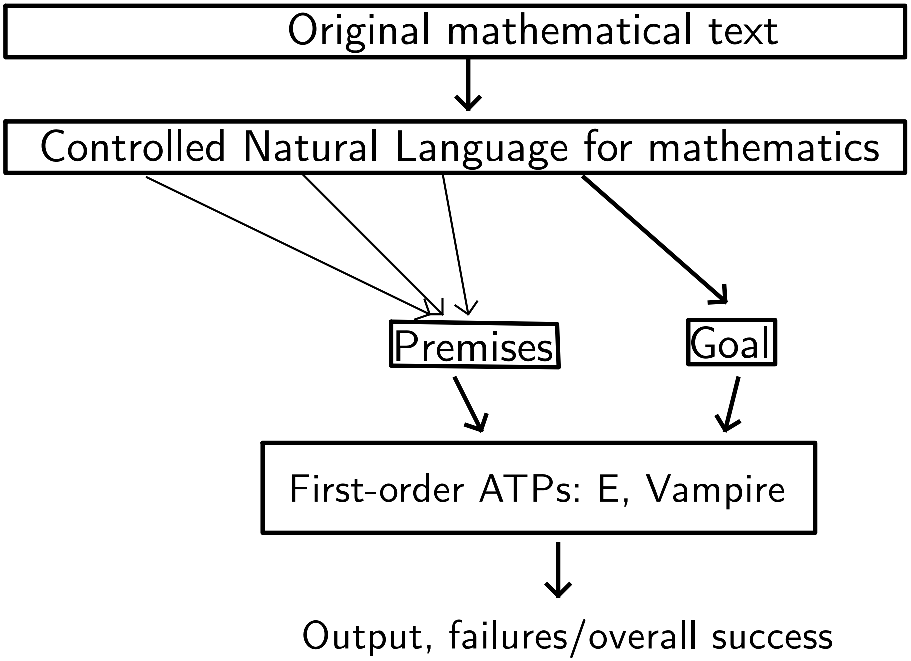
Naturalness of proof texts





Naturalness of proof texts

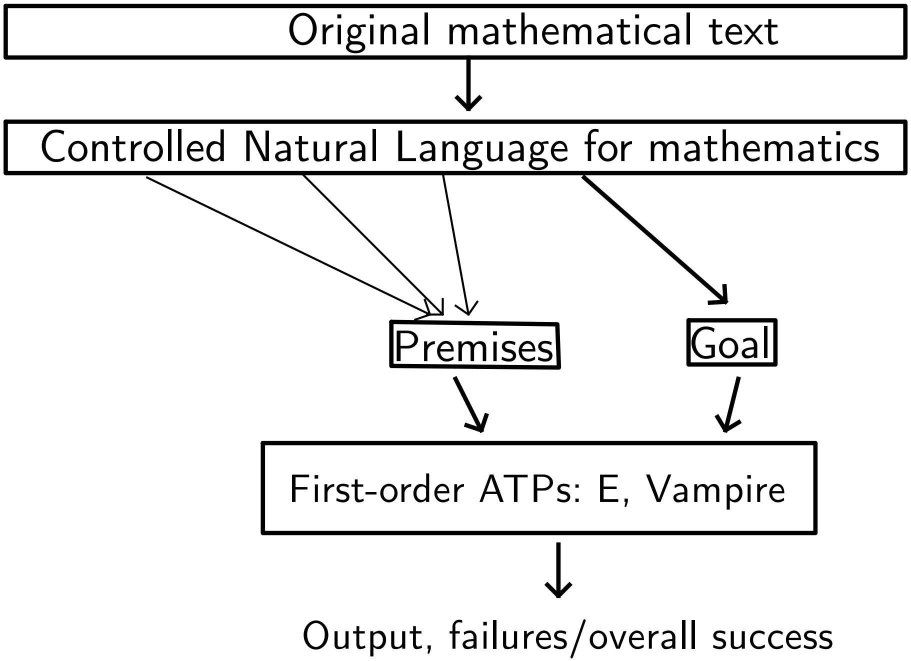
Controlled Natural Language ForTheL defined by a formal phrase structure grammar



Naturalness of proof texts

Controlled Natural Language ForTheL defined by a formal phrase structure grammar

Readable formalizations in LaTeX format, leveraging LaTeX macros



Naturalness of proof texts

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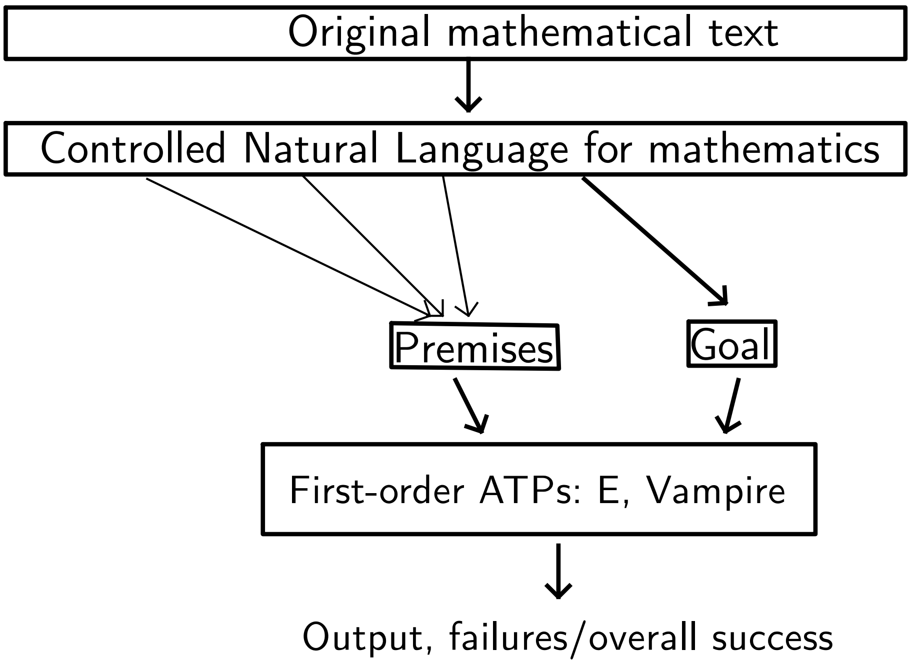
Readable formalizations in LaTeX format, leveraging LaTeX macros

Literate style, interleaving formalizations and explanatory text



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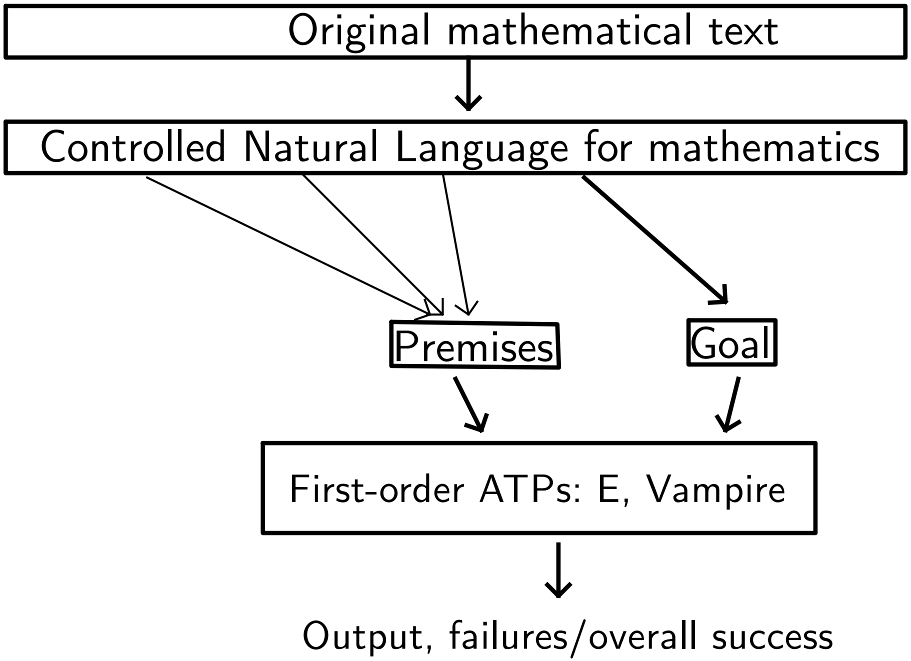
Literate style, interleaving formalizations and explanatory text

Strong ATPs to enable human-like proof steps



Naturalness of proof texts

Controlled Natural Language ForTheL defined by a formal phrase structure grammar

Readable formalizations in LaTeX format, leveraging LaTeX macros

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Strong ATPs to enable human-like proof steps

Embedded in the Isabelle Prover IDE for interactive editing and checking

The $\mathbb{N}$aproche formalization was motivated and guided by the Lean formalization of perfectoid *spaces* by Kevin Buzzard, Johan Commelin, and Patrick Massot.

```
-- We fix a prime number p
parameter (p : primes)

/-- A perfectoid ring is a Huber ring that is complete, uniform,
that has a pseudo-uniformizer whose p-th power divides p in the power bounded subring,
and such that Frobenius is a surjection on the reduction modulo p.-/
structure perfectoid_ring (R : Type) [Huber_ring R] extends Tate_ring R : Prop :=
(complete  : is_complete_hausdorff R)
(uniform   : is_uniform R)
(ramified  : ∃ ϖ : pseudo_uniformizer R, ϖ^p ∣ p in R°)
(Frobenius : surjective (Frob R°/p))

/-
CLVRS ("complete locally valued ringed space") is a category
whose objects are topological spaces with a sheaf of complete topological rings
and an equivalence class of valuation on each stalk, whose support is the unique
maximal ideal of the stalk; in Wedhorn's notes this category is called 𝒱.
A perfectoid space is an object of CLVRS which is locally isomorphic to Spa(A) with
A a perfectoid ring. Note however that CLVRS is a full subcategory of the category
`PreValuedRingedSpace` of topological spaces equipped with a presheaf of topological
rings and a valuation on each stalk, so the isomorphism can be checked in
PreValuedRingedSpace instead, which is what we do.
-/

/-- Condition for an object of CLVRS to be perfectoid: every point should have an open
neighbourhood isomorphic to Spa(A) for some perfectoid ring A.-/
def is_perfectoid (X : CLVRS) : Prop :=
∀ x : X, ∃ (U : opens X) (A : Huber_pair) [perfectoid_ring A],
  (x ∈ U) ∧ (Spa A ≅ U)

/-- The category of perfectoid spaces.-/
def PerfectoidSpace := {X : CLVRS // is_perfectoid X}

end
```

Boundedness in topological rings

Think of rings of formal power series like

$$R = S((X)) = \left\{ \sum_{n=N}^{\infty} a_n X^n, N \in \mathbb{Z} \right\}$$

with metric

$$d(f, g) = 2^{-\mathrm{ord}(f-g)}.$$

Let R denote a ring that is a topological space.

Definition 240 (title = L 42). Assume that B is a subset of R. B is bounded in R iff for all neighborhoods U of 0 in R there exists a neighborhood V of 0 in R such that $v \cdot b \in U$ where $v \in V$ and $b \in B$.

Definition 251 (title = L 179). Let r be an element of R. r is powerbounded in R iff $\{r^{n,R} \mid n \in \mathbb{N}\}$ is bounded in R.

Boundedness in topological rings

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$$R = S((X)) = \left\{ \sum_{n=N}^{\infty} a_n X^n, N \in \mathbb{Z} \right\}$$

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$\{\sum_{n=N_0}^{\infty} a_n X^n\}$ is bounded

Definition 251 (title = L 179). Let r be an element of R. r is powerbounded in R iff $\{r^{n,R} \mid n \in \mathbb{N}\}$ is bounded in R.

$\sum_{n=1}^{\infty} a_n X^n$ is power-bounded

Uniform and Huber rings

Definition 259 (title = L 310). $R^o = \{x \in R \mid x$ is power-bounded in $R\}$.

Definition 233. G is nonarchimedean iff every neighborhood U of 0^G in G has a subset S that is a subgroup of G and open in G.

Lemma 268 (title = L 371). Let R be nonarchimedean. Then R^o is a subring of R.

Definition 269 (title = 380). R is uniform iff R^o is a bounded subset of R.

Definition 276. A Huber ring is a topological ring R such that for some subset U of R and some finite subset T of U $\{U^{n,R} \mid n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of R and $T \cdot U = U \cdot U \subseteq U$.

Uniform and Huber rings

Definition 259 (title = L 310). $R^o = \{x \in R \mid x \text{ is power-bounded in } R\}$.

$R^o = \{\sum_{n=0}^{\infty} a_n X^n\}$

Definition 233. G is nonarchimedean iff every neighborhood U of 0^G in G has a subset S that is a subgroup of G and open in G.

$\{\sum_{n=N_0}^{\infty} a_n X^n\}$ is an open subgroup of 0.

Lemma 268 (title = L 371). Let R be nonarchimedean. Then R^o is a subring of R.

Definition 269 (title = 380). R is uniform iff R^o is a bounded subset of R.

Definition 276. A Huber ring is a topological ring R such that for some subset U of R and some finite subset T of U $\{U^{n,R} \mid n \in \mathbb{N}\}$ is a fundamental system of neighborhoods of R and $T \cdot U = U \cdot U \subseteq U$.

$U = \{\sum_{n=1}^{\infty} a_n X^n\}$

$T = \{X\}$

Definition 270 (title = L 30). Let r be an element of R. r is topologically nilpotent in R iff for all neighborhoods U of 0 in R there exists a natural number N such that $r^n \in U$ for all natural numbers n such that $n > N$.

Definition 279. A pseudouniformizer of R is a unit in R that is topologically nilpotent in R.

Definition 281. A Tate ring is a Huber ring that has a pseudouniformizer.

Tate rings

Definition 270 (title = L 30). Let r be an element of R. r is topologically nilpotent in R iff for all neighborhoods U of 0 in R there exists a natural number N such that $r^n \in U$ for all natural numbers n such that $n > N$.

X is topologically nilpotent

Definition 279. A pseudouniformizer of R is a unit in R that is topologically nilpotent in R.

X is a unit: $XX^{-1} = 1$

Definition 281. A Tate ring is a Huber ring that has a pseudouniformizer.

Perfectoid rings

Let R denote a Huber ring.

Signature 282. p is a prime number.

Definition 283. Let $x \in R$. $\Phi(x) = x^p$.

Definition 291. Let $S, T \subseteq R$. Let $a \in S$ and $b \in T$. $\Phi : S/a \cong T/b$ iff (for every $x, y \in S$ if $\Phi(x) \equiv_T \Phi(y)$ mod b within R then $x \equiv_S y$ mod a within R) and (for every $z \in T$ there exists $w \in S$ such that $z \equiv_T \Phi(w)$ mod b within R).

Let R denote a Tate ring.

Definition 293. R is perfectoid iff R is complete and uniform and there exists a pseudouniformizer ϖ of R such that $\varpi^p | p^{[R]}$ in R^o within R and

$$\Phi : R^o/\varpi \cong R^o/\varpi^p.$$

Lean **$\mathbb{N}$aproche**

```
/--A subset B of a topological ring
is bounded if for all neighbourhoods
U of 0 ∈ R, there exists a
neighbourhood V or 0 such that for
all v ∈ V and b ∈ B we have v*b ∈
U.

See [Wedhorn, Def 5.27, p. 36]. -/

def is_bounded (B : set R) : Prop :=
∀ U ∈ nhds (0 : R), ∃ V ∈ nhds (0 :
R), ∀ v ∈ V, ∀ b ∈ B, v*b ∈ U
```

Lean

```
/--A subset B of a topological ring
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See [Wedhorn, Def 5.27, p. 36]. -/

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R), ∀ v ∈ V, ∀ b ∈ B, v*b ∈ U
```

$\mathbb{N}$aproche

Let R denote a ring that is a topological space.

Definition. *Assume that B is a subset of R. B is bounded in R iff for all neighborhoods U of 0^R in R there exists a neighborhood V of 0^R in R such that $v \cdot^R b \in U$ where $v \in V$ and $b \in B$.*

Lean

```
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U of 0 ∈ R, there exists a
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See [Wedhorn, Def 5.27, p. 36]. -/

def is_bounded (B : set R) : Prop :=
∀ U ∈ nhds (0 : R), ∃ V ∈ nhds (0 :
R), ∀ v ∈ V, ∀ b ∈ B, v*b ∈ U
```

$\mathbb{N}$aproche

Let R denote a ring that is a topological space.

Definition. *Assume that B is a subset of R. B is bounded in R iff for all neighborhoods U of 0^R in R there exists a neighborhood V of 0^R in R such that $v \cdot^R b \in U$ where $v \in V$ and $b \in B$.*

Wedhorn original:

Definition 5.27. Let A be a topological ring. A subset B of A is called *bounded* if for every neighborhood U of 0 in A there exists an open neighborhood V of 0 in A such that $vb \in U$ for all $v \in V$ and $b \in B$.

Lean

```
/--A subset of a bounded subset
is bounded. See [Wedhorn, Rmk
5.28(2)].-/

lemma subset {S₁ S₂ : set R} (h
: S₁ ⊆ S₂) (H : is_bounded S₂) :
is_bounded S₁ :=

begin
intros U hU,
rcases H U hU with ⟨V, hV₁, hV₂⟩,
use [V, hV₁],
intros v hv b hb,
exact hV₂ _ hv _ (h hb),
end
```

$\mathbb{N}$aproche

Lemma 1. (title = L 136) *Every subset of every bounded subset of R is a bounded subset of R.*

Proof. Let B be a bounded subset of R. Let $A \subseteq B$. Let U be a neighborhood of 0^R in R. Take a neighborhood V of 0^R in R such that $V \star^R B \subseteq U$. Then $V \star^R A \subseteq V \star^R B \subseteq U$. $V \star^R A \subseteq U$. $\square$

Lean

```
  /--The sum of two power bounded
elements of a nonarchimedean ring is
power bounded.-/

lemma add (hR : nonarchimedean R)
(a b : R)(ha : is_power_bounded
a) (hb : is_power_bounded b) :
is_power_bounded (a + b) :=

begin

  rw singleton at ha hb ⊢,

  refine subset _ (add_group.closure
hR (union ha hb)),

  rw set.singleton_subset_iff,

  apply is_add_submonoid.add_mem;

    apply add_group.subset_closure;
simp

end
```

$\mathbb{N}$aproche

Lemma 2. (title = L 290) *Let R be nonarchimedean. Let a, b be elements of R that are powerbounded in R. Then $a +^R b$ is powerbounded in R.*

Proof. Let U be a neighborhood of 0^R in R. Take a subset U' of U that is a subgroup of R and open in R. U' is a neighborhood of 0^R in R. [timelimit 30] Take a neighborhood V of 0^R in R such that $v \cdot^R b^{n,R} \in U'$ where $v \in V$ and n is a natural number. [timelimit 30] Take a neighborhood W of 0^R in R such that $w \cdot^R a^{m,R} \in V$ where $w \in W$ and m is a natural number. [timelimit 3]
(1) $w \cdot^R (a^{m,R} \cdot^R b^{n,R}) \in U'$ where $w \in W$ and m, n are natural numbers.
Proof. Let $w \in W$ and m, n be natural numbers. $w \cdot^R a^{m,R} \in V$ and $w \cdot^R (a^{m,R} \cdot^R b^{n,R}) = (w \cdot^R a^{m,R}) \cdot^R b^{n,R} \in U$. qed. $\square$

......

Lean

```
import topology.basic

import topology.algebra.ring

import algebra.group_power

import ring_theory.subring

import tactic.ring


import for_mathlib.topological_rings

import
for_mathlib.nonarchimedean.adic_topology
```

ℕaproche

Contents

2 Natural Numbers

We introduce the notion (or type) of natural numbers. Together with an induction axiom to be stated later, the natural numbers can be understood as the inductive type generated by 0 and +1.

In this chapter we proceed towards prime numbers and divisibility properties of factorials and binomial coefficients.

2.1 Axioms

Signature 28. A natural number is a mathematical object.

Let n, m, k, l, i, j denote natural numbers.

Definition 29. $\mathbb{N}$ is the collection of natural numbers.

Axiom 30 (Axiom of Infinity). $\mathbb{N}$ is a set.

Signature 31. 0 is a natural number.

Let x is nonzero stand for $x \neq 0$.

Signature 32. 1 is a nonzero natural number.

Signature 33. $m + n$ is a natural number.

Axiom 34. If n is a nonzero natural number then $n = m + 1$ for

8

5 Rings

5.1 Axioms

We shall only consider commutative rings with 1. After defining a group as a *set* with further structure, we can now define a ring as a *group* together with multiplication and a 1.

Signature 130. A ring is an additive group.

Let R denote a ring.

Signature 131. 1^R is an element of R such that $1^R \neq 0^R$.

Signature 132. Let $x, y \in R$. $x \cdot^R y$ is an element of R.

Axiom 133. $(x \cdot^R y) \cdot^R z = x \cdot^R (y \cdot^R z)$ for all $x, y, z \in R$.

Axiom 134. $x \cdot^R 1^R = x$ for all $x \in R$.

Axiom 135 (title = Commutativity). $x \cdot^R y = y \cdot^R x$ for all $x, y \in R$.

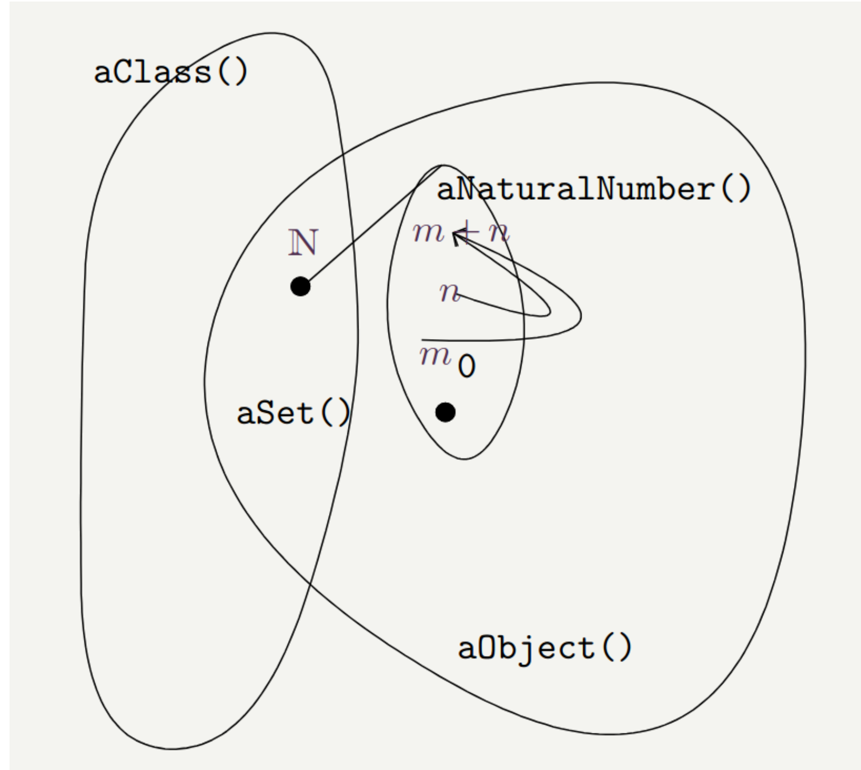
Axiom 136 (title = Distributivity). Let $x, y, z \in R$. $(x +^R y) \cdot^R z = (x \cdot^R z) +^R (y \cdot^R z)$.

Again readability is improved if we hide the recurring superscript ^{R} by the above method.

Preliminaries in the $\mathbb{N}$aproche formalization

The $\mathbb{N}$aproche preliminaries build a highly structured FO universe with FO-defined notions ($\sim$ types).

$\mathbb{N}$aproche provides rudimentary notions of *objects*, *sets* and *classes*, that can be further specified by axioms.



Lean

$\mathbb{N}$aproche

Lean

– original, comprehensive formalization

$\mathbb{N}$aproche

– formalization of a part of the Lean formalization

Lean

– original, comprehensive formalization

– unified Lean foundations (mathlib)

$\mathbb{N}$aproche

– formalization of a part of the Lean formalization

– ad hoc preliminaries

Lean

– original, comprehensive formalization

– unified Lean foundations (mathlib)

– within a big theory

$\mathbb{N}$aproche

– formalization of a part of the Lean formalization

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– “little theory”

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– formalization of a part of the Lean formalization

– ad hoc preliminaries

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– FOL with methods for first-order defined soft types

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– declarative proofs with implicit proof details

– heavy use of ATPs

- checking the perfectoid formalization takes $\sim$ 30 minutes

Lean

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– computer language

– explicit imperative tactic proofs

- efficient proof checking

Fully developed proving and programming language with continuous development, and growing support, libraries and user community

$\mathbb{N}$aproche

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Experimental, explorative proof of concept for Natural Language Proof Checking

Conclusions and future work

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– fully formal mathematics appears possible within a traditional ($\mathbb{N}$aproche-)style of mathematical writing

Conclusions and future work

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– can the translations and processings of various languages in natural formal mathematics be supported by machine learning and LLMs? Controlled Natural Languages like the $\mathbb{N}$aproche input language may be advantageous for LLMs since they are "natural languages".

Thank you!

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https://naproche.github.io/
https://github.com/naproche/FLib/blob/master/PerfectoidRings/perfectoidrings.ftl.tex

https://isabelle.in.tum.de/website-Isabelle2024/
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