A stronger solution to Hilbert‘s Tenth Problem, and its *in-situ* formalization

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The Problem (1900)

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$\Longleftrightarrow$

Set of squares

1 49 4

0

16 25 ...

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Hilbert Tenth's Problem over $\mathbb{Q}$?

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More specifically:

Hilbert Tenth's Problem for bounded complexity ?

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More specifically:

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DEF

$(\nu, \delta)_{\mathbb{N}}$ is a **universal pair** if any Diophantine set can be represented by a polynomial with i) at most ν unknowns in $\mathbb{N}$, ii) total degree at most δ.

Complicated Diophantine Equations

Is the set of primes Diophantine?

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Yes, primes are recursively enumerable. Explicitly:

$$
\begin{aligned}
(k+2)\Big\{1 &- [wz+h+j-q]^2 \\
&- [(gk+2g+k+1)(h+j)+h-z]^2 \\
&- [2n+p+q+z-e]^2 \\
&- \left[16(k+1)^3(k+2)(n+1)^2+1-f^2\right]^2 \\
&- \left[e^3(e+2)(a+1)^2+1-o^2\right]^2 \\
&- \left[(a^2-1)y^2+1-x^2\right]^2 \\
&- \left[16r^2y^4(a^2-1)+1-u^2\right]^2 \\
&- [n+l+v-y]^2 \\
&- \left[\left((a+u^2(u^2-a))^2-1\right)(n+4dy)^2+1-(x+cu)^2\right]^2 \\
&- \left[(a^2-1)l^2+1-m^2\right]^2 \\
&- \left[q+y(a-p-1)+s(2ap+2a-p^2-2p-2)-x\right]^2 \\
&- \left[z+pl(a-p)+t(2ap-p^2-1)-pm\right]^2 \\
&- [ai+k+1-l-i]^2 \\
&- \left[p+l(a-n-1)+b(2an+2a-n^2-2n-2)-m\right]^2\Big\}.
\end{aligned}
$$

Complicated Diophantine Equations

Collatz Problem

$$f(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ even} \\ 3n+1, & \text{else} \end{cases}$$

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⇒ 36-page polynomial

Collatz Conjecture:
This polynomial encodes the Diophantine set $\mathbb{N}$.

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Other "Diophantine" Problems

– Goldbach Conjecture

– ABC Conjecture

– Riemann Hypothesis

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? Given a Diophantine equation, can we find an equivalent, but simpler one?

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Universal Pairs over $\mathbb{N}$ and $\mathbb{Z}$

$$a \geq 3$$

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Four Squares Theorem:
Any $n \in \mathbb{N}$ is given by
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Basic translation of pairs:

$$(\nu, \delta)_{\mathbb{N}} \implies (4\nu, 2\delta)_{\mathbb{Z}}$$

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! *For an axiomatizable theory T and any proposition P, if P has a proof in T, then P has another proof consisting of 100 additions and multiplications of integers.*

[Jones 1982]

First Nontrivial Universal Pair in $\mathbb{Z}$

THM

Let $(\nu, \delta)_{\mathbb{N}}$ be universal. Then

$$\big(11, \eta(\nu, \delta)\big)_{\mathbb{Z}}$$

is universal, where

$$\eta(\nu, \delta) = 15\,616 + 233\,856\,\delta + 233\,952\,\delta\,(2\delta + 1)^{\nu+1} + 467\,712\,\delta^2\,(2\delta + 1)^{\nu+1}.$$

COR

The pair

$$(11, 1\,681\,043\,235\,226\,619\,916\,301\,182\,624\,511\,918\,527\,834\,137\,733\,707\,408\,448\,335\,539\,840)$$

$$\approx (11, 1.68105 \cdot 10^{63})$$

is universal.

Optimizing universality

$$\begin{aligned}
\mathfrak{b}(a,f) &:= 1 + 3(2a+1)f \\
\mathcal{B}(a,f) &:= \beta\,\mathfrak{b}^{\delta} \\
M(a,f) &:= \mathrm{mask}(\mathfrak{b}, \mathcal{B}, \mathbf{n}) \\
N_0(a,f) &:= \mathcal{B}^{(\delta+1)^{\nu}+1} \\
N_1(a,f) &:= 4\mathcal{B}^{(2\delta+1)(\delta+1)^{\nu}+1} \\
N(a,f) &:= N_0 N_1 \\
c(a,f,g) &:= 1 + a\mathcal{B} + g \\
\mathcal{K}(a,f,g) &:= \mathrm{value}(c, \mathcal{B}) \\
\mathcal{S}(a,f,g) &:= g + 2\mathcal{K} N_0 \\
\mathcal{T}(a,f) &:= M + (\mathcal{B}-2)\mathcal{B}^{(\delta+1)^{\nu+1}} N_0 \\
\mathcal{R}(a,f,g) &:= (\mathcal{S}+\mathcal{T}+1)N + \mathcal{T} + 1 \\
\mathcal{X}(a,f,g) &:= (N-1)\mathcal{R} \\
\mathcal{Y}(a,f) &:= N^2
\end{aligned}$$

$$\begin{aligned}
U &:= 2lXY \\
V &:= 4gwY \\
A &:= U(V+1) \\
B &:= 2X+1 \\
C &:= B + (A-2)h \\
D &:= (A^2-4)C^2+4 \\
E &:= C^2 Dx \\
F &:= 4(A^2-4)E^2+1 \\
G &:= 1 + CDF - 2(A+2)(A-2)^2E^2 \\
H &:= C + BF + (2y-1)CF \\
I &:= (G^2-1)H^2+1 \\
J &:= X + 1 + k(U^2V-2)
\end{aligned}$$

$$\begin{aligned}
&DFI \in \square \\
&(U^4V^2-4)J^2+4 \in \square \\
&(2A-5) \mid (3bwC - 2(b^2w^2-1)) \\
&\left(\frac{C}{J} - lY\right)^2 < \frac{1}{16g^2}\,.
\end{aligned}$$

$$\begin{aligned}
A_1 &:= \mathfrak{b} \\
A_2 &:= DFI \\
A_3 &:= (U^4V^2-4)J^2+4 \\
S &:= 2A-5 \\
T &:= 3\mathfrak{b}wC - 2(\mathfrak{b}^2w^2-1)
\end{aligned}$$

Any chance Isabelle could help…?

The General Strategy

Given: Universal Pair $(\nu, \delta)_{\mathbb{N}}$ and a Diophantine Set A

$$a \in A$$

$$\Updownarrow$$

$$\exists y_1, \ldots, y_\nu \in \mathbb{N}^\nu : P_A(a, y_1, \ldots, y_\nu) = 0$$
$$\text{where } \deg P_A \leq \delta$$

$$\Updownarrow$$

…intermediate representations…

$$\Updownarrow$$

$$\exists y_1, \ldots, y_{11} \in \mathbb{Z}^{11} : P(a, y_1, \ldots, y_{11}) = 0$$
$$\text{where } \deg P \leq \eta(\nu, \delta)$$

Project Organisation

Core Student Workgroup at ENS Paris

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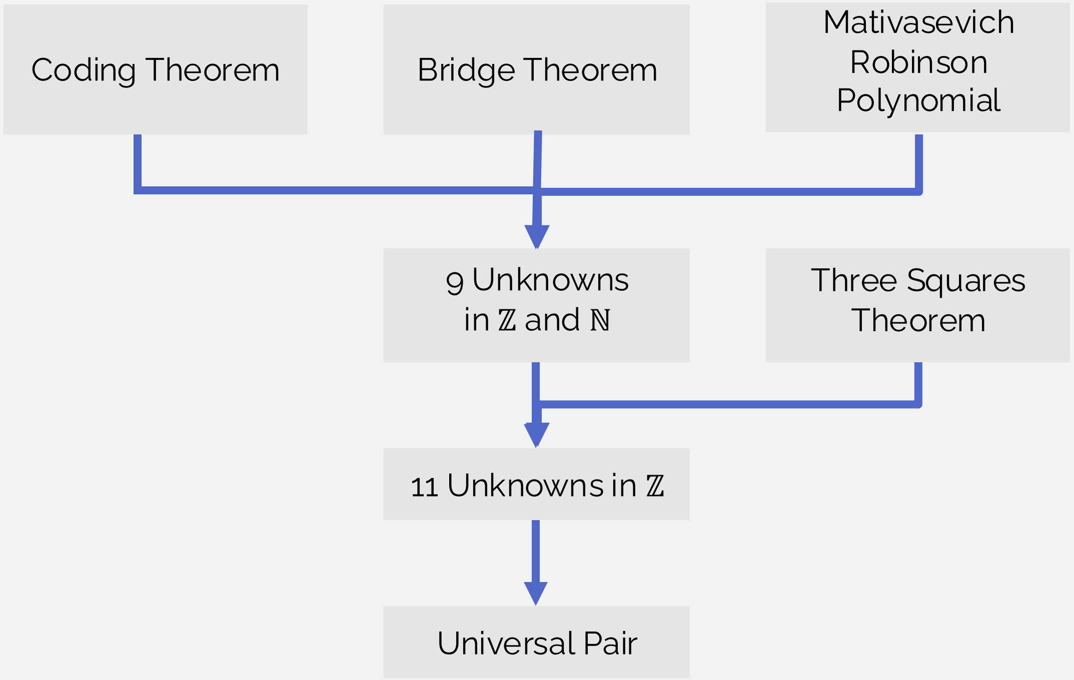
Structure of the mathematics emerged through formalisation



Polynomials in Isabelle

Isabelle datatype `mpoly` for multivariate polynomials

Example: `1 + 3 * (2 * Var 0 + 1) * Var 1`

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\end{aligned}$$

Command `poly_extract`

```
definition b :: "int ⇒ int ⇒ int" where
  "b a f ≡ 1 + 3*(2*a + 1) * f"

poly_extract b
```

```
consts
  b_poly :: "int mpoly"
Generated definition: 1 + 3 * (2 * Var 0 + 1) * Var 1
Proved correctness theorem: ⋀fn.
 insertion fn b_poly = coding_variables.b (fn 0) (fn 1)
```

Further command `poly_degree`

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Formalized in its natural environment: A maths department

Manuscript & Formalisation developed at the same time

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Benefits of the collaboration ITP <-> Researcher:

Fixing Bugs
Streamlining arguments
Precise Dependencies
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Isabelle works as a proof **assistant**

Future Work

Isabelle Feature Wishlist: Blueprint Tool

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Means continuing the mathematical research!

- $2 \leq \nu \leq 10$?
- Universal Pairs for multiple parameters

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Universal Pairs for multiple parameters

We unlock new research methods for researching extensions of Hilbert's Tenth Problem