Abstract, Compositional Consistency:

Isabelle/HOL Locales for Completeness à la Fitting

ITP '25

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Model existence for Natural Deduction (ND)

- We have a concrete calculus (natural deduction) for FOL with concrete proof rules:

$$\frac{\varphi \in \Gamma}{\Gamma \vdash \varphi}\text{Assm} \qquad \frac{\Gamma \vdash \varphi \quad \Gamma \vdash \psi}{\Gamma \vdash \varphi \wedge \psi}\wedge\text{I}$$

$$\frac{\Gamma \vdash \varphi \rightarrow \psi \quad \Gamma \vdash \varphi}{\Gamma \vdash \psi}\rightarrow\text{E} \qquad \frac{\Gamma \vdash \forall x.\ \varphi(x)}{\Gamma \vdash \varphi(t)}\forall\text{E}$$

...

- A formula set Γ is consistent wrt. ⊢ when we cannot derive a contradiction from it (i.e. $\neg(\Gamma \vdash \bot)$).
- The model existence theorem (for natural deduction):
  - Any ND-consistent set has a model.
  - $\neg(\Gamma \vdash \bot) \rightarrow \exists M.\ M \models \Gamma$
- From this follows completeness: Valid formulas are provable

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Understood conjunctively

This can be formalized!

- Model existence for natural deduction and other proof systems have been formalized in proof assistants using deep embeddings of FOL syntax.
  - E.g. Berghofer 2007 in Isabelle/HOL.
    Follows Melvin Fitting's 1996 textbook
    "First-order Logic and Automated Theorem Proving".
  - Many other fantastic results in this direction.
- Model existence theorems for other logics have also been formalized.
- In this work we provide a general framework for such model existence proofs (and more).

The plan for the talk

- I show you a concrete model existence proof for natural deduction and first-order logic.
  - Follows Melvin Fitting's 1996 textbook "First-order Logic and Automated Theorem Proving"
- I show you our generalization.
- I show you some instances.

A first generalization: Smullyan's uniform notation

- Characterize first-order logic with:
  - *Conjunctive*, *disjunctive*, *universal* and *existential* kinds.
  - Already generalizes from concrete FOL syntax actually.

α	$\varphi \wedge \psi : \alpha_1 = \varphi, \alpha_2 = \psi$	$\neg(\varphi \to \psi) : \alpha_1 = \varphi, \alpha_2 = \neg\psi$
β	$\varphi \to \psi : \beta_1 = \neg\varphi, \beta_2 = \psi$	$\neg(\varphi \wedge \psi) : \beta_1 = \neg\varphi, \beta_2 = \neg\psi$
γ	$\forall x.\, \varphi(x) : \gamma(t) = \varphi(t)$	$\neg(\exists x.\, \varphi(x)) : \gamma(t) = \neg\varphi(t)$
δ	$\exists x.\, \varphi(x) : \delta(t) = \varphi(t)$	$\neg(\forall x.\, \varphi(x)) : \delta(t) = \neg\varphi(t)$

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$\alpha : \land$

$\beta : \lor$

$\gamma : \forall$

$\delta : \exists$

Model existence for natural deduction for FOL

- Model existence
  - If S is consistent then S has a model.
- Proof idea
  1. Extend S to a maximal consistent set S'.
  2. S' is a Hintikka set.
  3. Therefore S' has a model M.
  4. M is also a model of S.

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  - What is a Hintikka set?
  - Why do these have models?
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Why do Hintikka sets have models?

$\alpha : \wedge$
$\beta : \vee$
$\gamma : \forall$
$\delta : \exists$

- A formula set S is a Hintikka set when:

conflict for all p, not both $p \in S$ and $\neg p \in S$

banned $\bot \notin S$ (and $\neg\top \notin S$)

double neg. if $\neg\neg\varphi \in S$ then $\varphi \in$ S

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \subseteq S$

beta if $\beta \in S$ then $\beta_1 \in S$ or $\beta_2 \in S$

gamma if $\gamma \in S$ then $\gamma(t) \in S$ for every (closed term) t **(…)**

delta$_{\mathbf{E}}$ if $\delta \in S$ then $\delta(a) \in S$ for *some* a (…)

Hintikka's lemma:
Any Hintikka set has a model.

(Why? Brief and vague explanation:
We can think of "is member of S" as "is true in the model we want to find". Then S will provide us with a Herbrand model.)

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What is a maximal consistent set?

- $C_{ND} = \{S \mid \neg(S \vdash \bot)\}$ -- set of all ND-consistent sets
  $S \in C_{ND}$ means "S is ND-consistent"

- mcs $S \longleftrightarrow S \in C_{ND} \wedge (\forall S' \in C_{ND}.\ S \subseteq S' \longrightarrow S = S')$

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Why are maximal consistent sets also Hintikka sets?

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- Some ND-consistency lemmas
- Generally they claim the existence of a bigger consistent set

  For any $S \in C_{ND}$

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- Informally:
  - Take S.
  - Enumerate through the universe of formulas, and add those to S that do not introduce inconsistency.

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- (This slide is a *very* simplified account of how and why this works.)

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For any $S \in C_{ND}$

Important!

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Fantastic!
We have now seen model existence for natural deduction!

A generalization by Smullyan

- Actually the Important! properties on the previous slide do not only hold for C_{ND}
- If we replace C_{ND} with one of the following then all the properties still hold:
- $C_{AxS} = \{S \mid \neg(S \vdash_{\mathrm{AxS}} \bot)\}$ where AxS is axiomatic system.
  - So we can get model existence for axiomatic system.
- $C_{comp} = \{S \mid$ all finite subsets of S are satisfiable$\}$
  - So we can get compactness
- ...
- There are more examples.
  - E.g. we can also get a weak downward Löwenheim-Skolem and Craig's interpolation theorem.

A generalization by Smullyan

$\alpha : \wedge$
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- A useful concept:
  - We call any set C a consistency property if it has the properties!

Important!

C is a consistency property if:
For any $S \in C$

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Abstract model existence theorem

- If C is a consistency property and $S \in C$ then S has a model.

Abstract model existence theorem

- If C is a consistency property and $S \in C$ then S has a model.

- Nice! To show that some property C ensures models of its members we only need to show that it is a consistency property!

Questions

- Does this idea work for other logics than FOL?
- Yes. (E.g. Fitting used it for modal logic and intuitionistic logic)

- Are the applications of the idea similar enough that we can make a general framework?
- Yes. (This work)

- Can such framework be expressed with locales in Isabelle/HOL?
- Yes. (This work)

- Can the locales really help prove formalize existence theorems for some concrete logics?
- Yes, bounded FOL, SOL, very recent modal logic. (This work)

The main idea: This slide is sliced into kinds

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delta$_E$ if $\delta \in S$ then $\delta(a) \in S$
for *some* a (…)

The main idea: This slide is sliced into kinds

- C is a consistency property if
  For any S $\in C$

conflict for all p,
not both $p \in S$ and $\neg p \in S$

banned $\bot \notin S$ (and $\neg\top \notin S$)

double neg. if $\neg\neg\varphi \in S$
then $\{\varphi\} \cup S \in C$

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in$

beta if $\beta \in S$ then $\{\beta_1\} \cup S \in C$
or $\{\beta_2\} \cup S \in C$

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- A formula set S is Hintikka when:

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Beta kind

if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \subseteq S$

beta if $\beta \in S$ then $\beta_1 \in S$
or $\beta_2 \in S$

gamma if $\gamma \in S$ then $\gamma(t) \in S$
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delta$_E$ if $\delta \in S$ then $\{\delta(a)\} \cup S \in C$
for *some* a (…)

Gamma kind

- A formula set S is Hintikka when:

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not both $p \in S$ and $\neg p \in S$

banned $\bot \notin S$ (and $\neg\top \notin S$)

double neg. if $\neg\neg\varphi \in S$
then $\varphi \in$ S

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \subseteq S$

beta if $\beta \in S$ then $\beta_1 \in S$
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Delta kind

delta$_{\mathbf{E}}$ if $\delta \in S$ then $\{\delta(a)\} \cup S \in C$
for *some* a (…)

- A formula set S is Hintikka when:

conflict for all p,
not both $p \in S$ and $\neg p \in S$

banned $\bot \notin S$ (and $\neg\top \notin S$)

double neg. if $\neg\neg\varphi \in S$
then $\varphi \in$ S

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \subseteq S$

beta if $\beta \in S$ then $\beta_1 \in S$
or $\beta_2 \in S$

ma if $\gamma \in S$ then $\gamma(t) \in S$
for every (closed term) t (…)

delta$_{\mathbf{E}}$ if $\delta \in S$ then $\delta(a) \in S$
for *some* a (…)

The main idea: This slide is sliced into kinds

- C is a consistency property if
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The framework

```
datatype ('x, 'fm) kind
  = Cond ‹'fm list ⇒ ('fm set set ⇒ 'fm set ⇒ bool) ⇒ bool› ‹'fm set ⇒ bool›
```

The framework

An Important! property

A corresponding part of the Hintikka definition.

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datatype ('x, 'fm) kind
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```

e.g.

e.g.

Alpha kind

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in C_{ND}$	**alpha** if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \subseteq S$

The framework

```
datatype ('x, 'fm) kind
  = Cond ‹'fm list ⇒ ('fm set set ⇒ 'fm set ⇒ bool) ⇒ bool› ‹'fm set ⇒ bool›
  | Wits ‹'fm ⇒ 'x ⇒ 'fm list›
```

One more constructor actually
-- essentially for δ formulas.

This needs to be handled differently
in the mcs construction.
I skipped this in my simplified explanation.

A locale for Kinds!

Now we introduce locale for Consistency Kinds

```
locale Consistency_Kind = Params map_fm params_fm
 for map_fm :: ‹('x ⇒ 'x) ⇒ 'fm ⇒ 'fm›
 and params_fm :: ‹'fm ⇒ 'x set› +
 fixes K :: ‹('x, 'fm) kind›
 assumes hintikka:
  ‹⋀C S. sat_E K C ⟹ S ∈ C ⟹ maximal C S ⟹ sat_H K S›
```

A locale for Kinds!

locale for parameter substitutions.

Now we introduce locale for Consistency Kinds

```
locale Consistency_Kind = Params map_fm params_fm
 for map_fm :: ‹('x ⇒ 'x) ⇒ 'fm ⇒ 'fm›
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 fixes K :: ‹('x, 'fm) kind›
 assumes hintikka:
  ‹⋀C S. sat_E K C ⟹ S ∈ C ⟹ maximal C S ⟹ sat_H K S›
```

This essentially says that e.g.

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \cup S \in C$

ensures

alpha if $\alpha \in S$ then $\{\alpha_1, \alpha_2\} \subseteq S$ on maximality.

But! We are here talking about only one kind (e.g. alpha).

A locale for Kinds!

Now we introduce locale for Consistency Kinds

```
locale Consistency_Kind = Params map_fm params_fm
 for map_fm :: ‹('x ⇒ 'x) ⇒ 'fm ⇒ 'fm›
 and params_fm :: ‹'fm ⇒ 'x set› +
 fixes K :: ‹('x, 'fm) kind›
 assumes hintikka:
  ‹⋀C S. sat⇩E K C ⟹ S ∈ C ⟹ maximal C S ⟹ sat⇩H K S›
```

A locale for Kinds!

Now we introduce locale for Consistency Kinds

```
locale Consistency_Kind = Params map_fm params_fm
 for map_fm :: ‹('x ⇒ 'x) ⇒ 'fm ⇒ 'fm›
 and params_fm :: ‹'fm ⇒ 'x set› +
 fixes K :: ‹('x, 'fm) kind›
 assumes hintikka:
  ‹⋀C S. sat_E K C ⟹ S ∈ C ⟹ maximal C S ⟹ sat_H K S›
 and respects_close:
  ‹⋀C. sat_E K C ⟹ sat_E K (close C)›
 and respects_alt:
  ‹⋀C. sat_E K C ⟹ subset_closed C ⟹ sat_A K
                    (mk_alt_consistency C)›
 and respects_fin:
  ‹⋀C. subset_closed C ⟹ sat_A K C ⟹ sat_A K
                                   (mk_finite_char C)›
```

Important for the MCS construction

More locales

- We have defined locales for alpha, beta, gamma, delta etc.
- We have shown them to specialize the `Consistency_Kind` locale.

Pre-Defined Kinds

- For a user-given predicate ↝ we can define the following:
  - (Under some natural conditions on each ↝.)

Alpha `‹ps ↝α qs ⟹ condα ps`

`(λC S. set qs ∪ S ∈ C)›`

Beta `‹ps ↝β qs ⟹ condβ ps`

`(λC S. ∃q ∈ set qs. {q} ∪ S ∈ C)›`

Gamma `‹ps ↝γ (F, qs) ⟹ condγ ps`

`(λC S. ∀t ∈ F S. set (qs t) ∪ S ∈ C)›`

...

- Likewise we can define `hintα`, `hintβ`, `hintγ`,...
- And then we have kinds:
  - `Cond condα hintα`
  - `Cond condβ hintβ`
  - `Cond condγ hintγ` ...

Combining Kinds

- We have seen kinds.
- But to get a definition of consistency property and Hintikka set we need to combine them.
- We have a locale for that.

```
locale Consistency_Kinds = Params map_fm params_fm
  for
    map_fm :: ‹('x ⇒ 'x) ⇒ 'fm ⇒ 'fm› and
    params_fm :: ‹'fm ⇒ 'x set› +
  fixes Ks :: ‹('x, 'fm) kind list›
  assumes all_kinds: ‹⋀K. K ∈ set Ks ⟹
                        Consistency_Kind map_fm params_fm K›
```

Combining Kinds

- The main theorem:
  - Consistent sets of formulas can be *extended* to maximal consistent sets, and these are *Hintikka*.

```
lemma mk_mcs_hintikka:
  assumes ‹propE Ks C› ‹S ∈ C› ‹enough_new S›
  shows ‹propH Ks (mk_mcs C S)›
```

- Here we have combined the individual consistency requirements into an "is consistency property set" definition (`propE Ks`)
- We have combined the individual Hintikka requirements into an "is Hintikka set" definition (`propH Ks`)
- And, shown that your formula set can be extended to be Hintikka.

Application:
“Bounded” First-Order Logic

Restricted Instantiation

- Consider first-order logic with the following rule:

$$\frac{\Gamma \vdash \forall x.\ \varphi(x) \qquad t \text{ is a sub-term of } \Gamma, \varphi}{\Gamma \vdash \varphi(t)}\ \forall\mathrm{E}$$

- Make use of the ability to bound our **gamma** kind:
  - ‹[⊥] ⤳ₓ [⊥]›
  - ‹[¬ (·P ts)] ⤳ₓ [·P ts]›
  - ‹[¬ (p ⟶ q)] ⤳$_α$ [p, ¬ q]›
  - ‹[p ⟶ q] ⤳$_β$ [¬ p, q]›
  - ‹[∀p] ⤳$_γ$ (λS. terms S, λt. [⟨t⟩p])›
  - ‹δ (¬ ∀p) x = [¬ ⟨⋆x⟩p]›

Application:
Second-Order Logic

Scaling Up

- Quantify over functions and predicates besides terms.
- **gamma**s for different quantifiers at different types:
  - ‹[∀p] $\leadsto_\gamma$ (λt. [⟨t/0⟩p])›
  - ‹[$\forall_P$ p] $\leadsto_{\gamma P}$ (λs. [$\langle s/0\rangle_P$ p])›
  - ‹[$\forall_F$ p] $\leadsto_{\gamma F}$ (λs. [$\langle s/0\rangle_F$ p])›
- Each **gamma** can only instantiate with one type of term
  - compose our consistency property of multiple **gamma**s.
- Mechanized completeness as before.

Application: Prior's Ideal Language
A very recent modal logic

A very recent modal logic

- Based on work by Blackburn, Braüner and Kofod.
- A very recent modal logic with Kripke semantics, and propositional quantification.
- See our paper, our formalization and the paper by Blackburn, Braüner and Kofod.

Conclusion

Conclusion

- Consistency properties provide an interface for building MCSs.
- An advantage of our framework is *modularity* and *locality*:
  - You prove correspondence between maximality and Hintikka "locally" for each Kind.
  - For the alpha, beta, gamma, delta we did it already.
  - So you can focus on the the syntax that makes your logic special!
- I hope you will prove model existence and completeness for your favorite logic with our framework :-D

Thank you!

Bonus slide!

Concrete Maximal Consistency

- A consistent set Γ is a maximally consistent set (MCS) when it contains every formula consistent with it:

$$\text{if } \Gamma \subseteq \Delta \text{ and } \Delta \text{ consistent, then } \Gamma = \Delta$$

- We can build an MCS by trying to add every formula and taking the union $\Delta = \cup_i \Delta_i$ (Lindenbaum-Tarski):

$$\Delta_0 = \Gamma$$

$$\Delta_{i+1} = \{\varphi_i, \psi(a)\} \cup \Delta_i \quad \text{if consistent and } \varphi_i = \exists x.\, \psi(x)$$

$$\Delta_{i+1} = \{\varphi_i\} \cup \Delta_i \quad \text{otherwise if consistent}$$

$$\Delta_{i+1} = \Delta_i \quad \text{otherwise}$$

Maximal Element?

- Set theory: under the axiom of choice, *finite character* of a family of sets C guarantees a maximal member wrt. ⊆:

```
‹finite_char C ≡
  ∀S. S ∈ C ⟷ (∀S' ⊆ S. finite S' ⟶ S' ∈ C)›
```

- Problem: imposing finite character might break $\textbf{delta}_{\mathbf{E}}$.
  - Exercise for the reader.

- Solution: interpret it universally rather than existentially.

$\textbf{delta}_{\mathbf{A}}$ if $\delta \in S$ then $\{\delta(a)\} \cup S \in C$ for every *new* a (…)

- How do we recover $\textbf{delta}_{\mathbf{E}}$? Manually!
  - As earlier in the Lindenbaum-Tarski construction.