Finiteness of Symbolic Derivatives in Lean

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Introduction

- *Brzozowski derivatives* of regular expressions

$$\mathcal{L}(der(c, R)) := \{w \in \Sigma^* \mid c \cdot w \in \mathcal{L}(R)\}$$
$$null(R) := \epsilon \in \mathcal{L}(R)$$

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- (Brzozowski, 64): *finiteness* of all iterated derivatives

$$\mathcal{D}er(R) = \{der_w^*(R) \mid w \in \boldsymbol{\sigma}^*\}/_{\cong}$$

quotiented by a relation called *ACI*-similarity:

$$\begin{array}{rcll} (L \Cup R) \Cup S & \cong & L \Cup (R \Cup S) & \text{Associativity} \\ L \Cup (R \Cup L) & \cong & L \Cup R & \text{Commutativity} \\ R \Cup R & \cong & R & \text{Idempotence} \end{array}$$

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  - `"abab"` vs `"abab"`

Regular Expressions with Lookarounds

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\begin{aligned}
R, S \;::=\; \psi \in \alpha \;\mid\; \varepsilon \mid\; R \Cup S \mid\; R \Cap S \mid\; R \cdot S \mid\; R^* \mid\; {\sim}R \\
\mid\; \texttt{(?=}R\texttt{)} \mid\; \texttt{(?<=}R\texttt{)} \mid\; \texttt{(?!}R\texttt{)} \mid\; \texttt{(?<!}R\texttt{)}
\end{aligned}
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- We work modulo an alphabet theory $\mathcal{A} = (\Sigma, \boldsymbol{\alpha}, \vDash, \bot, \top, \sqcup, \sqcap, ^{\mathsf{c}})$
  For example, $\psi_{upper} \in \alpha$ and $[\![\psi_{upper}]\!] = [A - Z]$

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  For example, $\psi_{upper} \in \alpha$ and $[\![\psi_{upper}]\!] = [A - Z]$
- Positive lookahead $\texttt{(?=}R\texttt{)}$ and lookbehind $\texttt{(?<=}R\texttt{)}$
  Negative lookahead $\texttt{(?!}R\texttt{)}$ and lookbehind $\texttt{(?<!}R\texttt{)}$

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Semantics

$$(xs, ys, zs) \models \texttt{(?=}R\texttt{)} \iff ys = \epsilon \wedge (xs, zs, \epsilon) \models R{\cdot}\top*$$

Example: $R = \texttt{(?=}\psi_{upper}\texttt{)}$ and `s = "aAbc"`

Then $(\texttt{"a"}, \epsilon, \texttt{"Abc"}) \models \texttt{(?=}\psi_{upper}\texttt{)}$

since $(\texttt{"a"}, \texttt{"Abc"}, \epsilon) \models \psi_{upper} \cdot \top*$ is a valid *future* match

Transition terms and symbolic derivatives

A symbolic derivative is a transition term i.e. trees of regexes

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inductive TTerm (α : Type) : Type where
| Leaf : RE α → TTerm α
| Node : RE α → TTerm α → TTerm α → TTerm α
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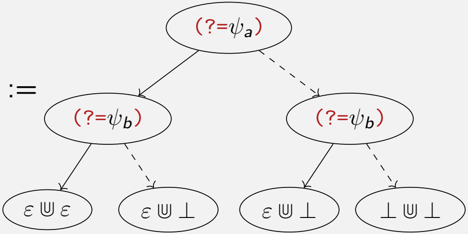
Example

Let ψ_a and ψ_b be atomic predicates.

$$\delta\ (\psi_a \Cup \psi_b) : \text{TTerm } \alpha :=$$



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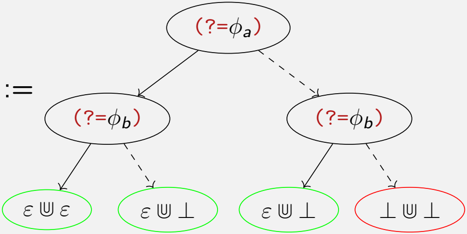

Semantics of transition terms

- Transition term $\approx$ a function from locations to regexes
- ... but it postpones all the nullability tests
- We define the evaluation function of type
  `Loc` σ `→ TTerm` α `→ RE` α

$$L[x] := L \quad (R, f, g)[x] := \begin{cases} f[x], \text{ if } \textit{null } R\ x; \\ g[x], \text{ otherwise.} \end{cases}$$

Symbolic derivatives

Let $\ell \in \mathsf{LA}$, $\psi \in \boldsymbol{\alpha}$ then δ : $\texttt{RE}\ \alpha \to \texttt{TTerm}\ \alpha$

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\delta\ \varepsilon &:= \bot & \delta\ \ell &:= \bot \\
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\delta\ (L \Cap R) &:= \delta\ L \Cap \delta\ R & \delta\ ({\sim}R) &:= {\sim}(\delta\ R) \\
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Theorem 1. $\forall x \in \mathsf{Loc}, R \in RE\ \alpha : (\delta\ R)[x] = der\ R\ x$

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Theorem 1. $\forall x \in \mathsf{Loc}, R \in RE\ \alpha : (\delta\ R)[x] = der\ R\ x$

$\rightarrow$ from now on, we can just work with the symbolic definition.

Iterated derivatives

- We can now compute the immediate derivatives of R:

  ```
  lvs :  TTerm α → RE α
  step (R : RE α) :  List (RE α) := lvs (δ R)
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\mathsf{step}\ (R{*}) &= \mathsf{step}\ R \cdot R{*}
\end{aligned}
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- We compute the n-th derivatives (words of length n):

  `steps : RE` α `→ Nat → List (RE` α`)`

Finiteness of the state space

- **Classical approach** (Brzozowski/DFA construction)
  - $\mathcal{D}er(R) = \{der_w(R) \mid w \in \Sigma^*\}/_{\cong}$
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- **Our approach**
  - Follow Antimirov's strategy for finiteness
  - While dealing with the extended class of expressions

Similarity

We define helpers to reason *up-to* a relation R

- List membership
  `x` $\in$`[ R ] ys` $:=\exists$ `y, R x y` $\wedge$ `y` $\in$ `ys`
- List inclusion
  `xs` $\subseteq$`[ R ] ys` $:=\forall$ `x` $\in$ `xs, x` $\in$`[ R ] ys`
- List equality
  `xs =[ R ] ys` $:=$ `xs` $\subseteq$`[ R ] ys` $\wedge$ `ys` $\subseteq$`[ R ] xs`

Similarity

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- List membership
  `x ∈[ R ] ys` $:=$ `∃ y, R x y ∧ y ∈ ys`
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- List equality
  `xs =[ R ] ys` $:=$ `xs ⊆[ R ] ys ∧ ys ⊆[ R ] xs`

Our relation: the ADI-similarity relation used for quotienting:

$(L \Cup R) \Cup S \cong L \Cup (R \Cup S)$	Associativity
$L \Cup (R \Cup L) \cong L \Cup R$	right Deduplication
$R \Cup R \cong R$	Idempotence

Finiteness of Antimirov derivatives

- Why is proving finiteness easy for Antimirov derivatives?

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\begin{array}{rcl}
\mathit{support}(\bot) & := & \emptyset \\
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- ACI is built into the set representation
- Can we use a similar strategy for Brzozowski-style derivatives?

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\begin{tikzcd}
a \Cup b \cdot c \arrow[r, "\mathrm{der}_a"] \arrow[d, "\texttt{pieces}"'] & \varepsilon \Cup \bot \cdot c \arrow[dl, "\ni"] \\
{[\bot, \varepsilon, a] \mathbin{+\!\!+} [\bot, \varepsilon, c, \bot \cdot c, \varepsilon \cdot c, b \cdot c]} &
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- **Key idea:** all derivatives can be given as union of `pieces`

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- We don't have commutativity of union so we have to consider all permutations of a list:
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```
def pieces : RE α → List (RE α)
  | ε       => [ε, Pred ⊥]
  | Pred φ  => [Pred φ, ε, Pred ⊥]
  | ?= r    => [?= r, ε, Pred ⊥] | ...
  | l ⋓ r   => pieces l ++ pieces r
  | l ⋒ r   => productWith (· ⋒ ·) ⊕(pieces l) ⊕(pieces r)
  | ~r      => map (~ ·) ⊕(pieces r)
  | l · r   => map (· · r) ⊕(pieces l) ++ pieces r
  | r*      => r* :: map (· · r*) ⊕(pieces r)
```

Main theorem

1. **Reflexivity**:
```
∀ r,
∃ xs, toSum xs ≅ r ∧ xs ∈ neSublists (pieces r)
```
2. **Transitivity**:
```
   e ∈ pieces f
 → f ∈ pieces g
 → e ∈[ (· ≅ ·) ] pieces g
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3. **One-step reconstruction**:
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∀ r d, d ∈ step r
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- The witness is `xs := ⊕(pieces R)`

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- We instead compute a finite overapproximation of all reachable derivatives
- (Moreira et al., 2012) avoid the need for normalisation modulo ACI by using Antimirov derivatives
- We take inspiration from the Antimirov finiteness proof, but adapt it to handle intersection and negation

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Allowed simplifications

```
⊥ ⋓ s ⇝ s     ~⊥ ⋓ s ⇝ ~⊥     ε · s ⇝ s
r ⋓ ⊥ ⇝ r     r ⋓ ~⊥ ⇝ ~⊥     r · ⊥ ⇝ ⊥

(r ⋓ s) · t ⇝ r · t ⋓ s · t
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- `r · s ⇝ s` and `r ⋓ s ⇝ r` are allowed

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Allowed simplifications

$$\bot \Cup s \leadsto s \qquad {\sim}\bot \Cup s \leadsto {\sim}\bot \qquad \varepsilon \cdot s \leadsto s$$
$$r \Cup \bot \leadsto r \qquad r \Cup {\sim}\bot \leadsto {\sim}\bot \qquad r \cdot \bot \leadsto \bot$$
$$(r \Cup s) \cdot t \leadsto r \cdot t \Cup s \cdot t$$

- $r \cdot s \leadsto s$ and $r \Cup s \leadsto r$ are allowed
- $r \cdot s \leadsto r$ is **not** allowed

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Thank you!

`github.com/ezhuchko/finiteness-derivatives`