Formalizing the Hidden Number Problem in Isabelle/HOL

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Joint work with Eric Ren and Katherine Kosaian

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- Formalization details

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- Yes! Idea: compute g^{ab} given a bit-leaking oracle.

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  $= \mathrm{MSB}_k(\alpha t)$
  $=: \mathcal{O}(t)$
- Main theorem: we can recover α given $\mathcal{O}$.

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- $961 < n := \lceil \log(p) \rceil$
- $d := 2 \cdot \lceil \sqrt{n} \rceil$
- $k := \lceil \sqrt{n} \rceil + \lceil \log(n) \rceil$

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- Idea: Approximate α using lattice methods.

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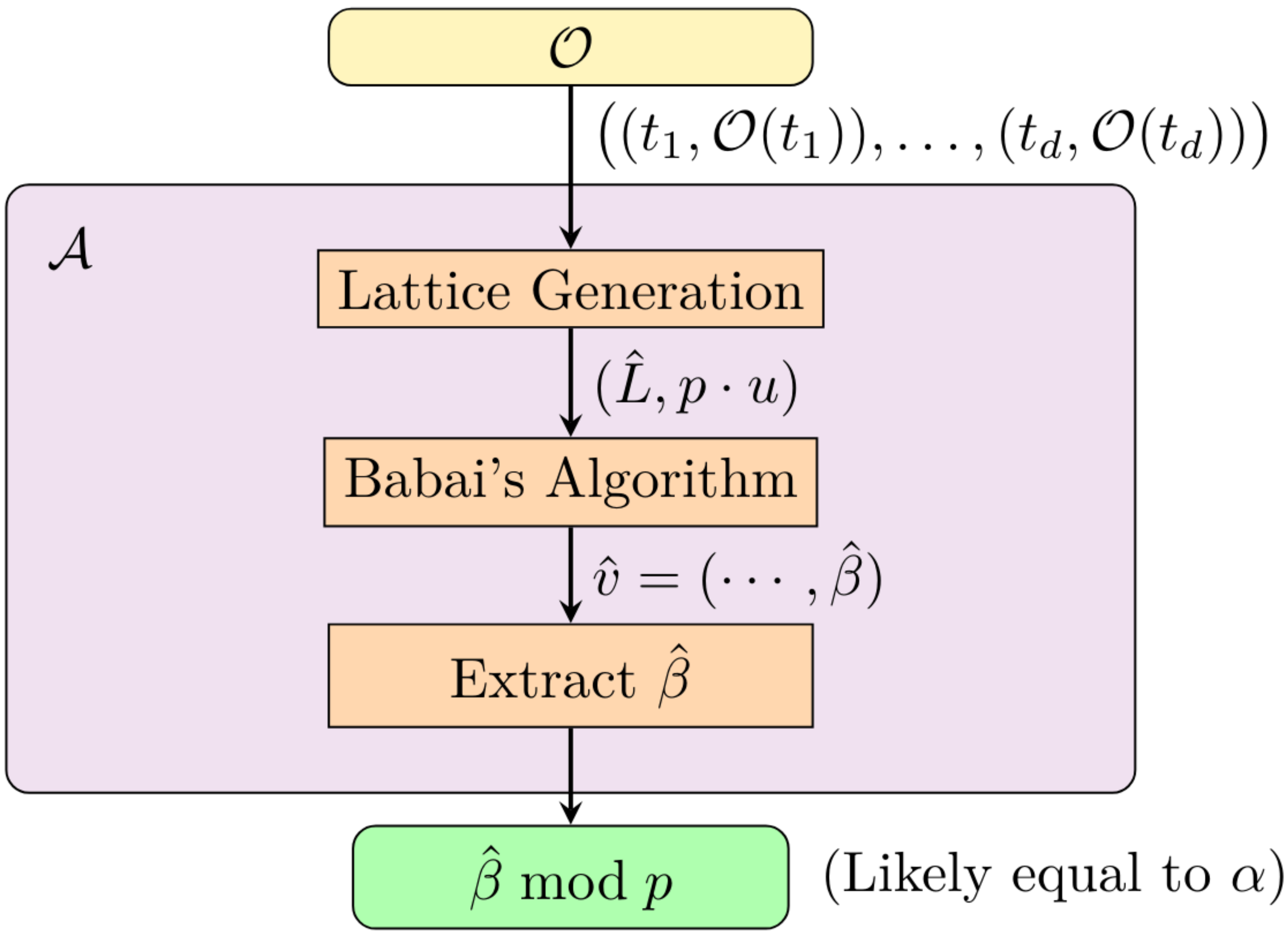
  - With Babai's nearest plane algorithm.
  - Finds $v \in L$ with $|u - v|$ close to optimal.

Adversary



$\mathcal{O}$
$((t_1, \mathcal{O}(t_1)), \ldots, (t_d, \mathcal{O}(t_d)))$
$\mathcal{A}$
Lattice Generation
$(\hat{L}, p \cdot u)$
Babai's Algorithm
$\hat{v} = (\cdots, \hat{\beta})$
Extract $\hat{\beta}$
$\hat{\beta} \bmod p$
(Likely equal to α)

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  - Isabelle/HOL automation very valuable.

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- We formalize $\sqrt{\dim(L)}(4/3)^{\dim(L)/2} D$, which suffices.

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- Intuitively true (lattice is discrete); annoying to formalize

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  - Can be removed since $\epsilon > 0$ is arbitrary.

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- Like LLL, our formal Babai's algorithm is executable.

Formalizing the HNP

Formalizing the HNP

Proof. Let β, γ be two integers. Define the *modular distance* between β and γ as

$$\mathrm{dist}_p(\beta,\gamma) = \min_{b\in\mathbb{Z}} |\beta - \gamma - bp|$$

For example, $\mathrm{dist}_p(1,p) = 1$. Suppose $\beta \not\equiv \gamma \pmod p$ and they are both integers in the range $[1, p-1]$. Define

$$A = \Pr_t\left[\mathrm{dist}_p(\beta t, \gamma t) > 2p/2^\mu\right]$$

where t is an integer chosen uniformly at random in $[1, p-1]$. Then

$$A = \Pr_t\left[\frac{2p}{2^\mu} < (\beta-\gamma)t \bmod p < p - \frac{2p}{2^\mu}\right] = \frac{\lfloor p - \frac{2p}{2^\mu}\rfloor - \lceil \frac{2p}{2^\mu}\rceil}{p-1} \geq 1 - \frac{5}{2^\mu}$$

This follows since for every $x \in [\frac{2p}{2^\mu}, p - \frac{2p}{2^\mu}]$ there exists a t such that $(\beta-\gamma)t = x \pmod p$. In general, a lattice point v has the form

$$v = (\beta t_1 - b_1 p,\ \beta t_2 - b_2 p, \ldots, \beta t_d - b_d p,\ \beta/p)$$

for some integers $\beta, b_1, \ldots, b_d$. Suppose $\| v - u \| < p/2^\mu$. We show that with probability at least $\frac{1}{2}$ the vector v satisfies $\beta \equiv \alpha \pmod p$ and $\beta t_i - b_i p \in [0, p]$ for all i. Observe that if $\beta = \alpha \pmod p$, then $\beta t_i - b_i p \in [0,p]$ for all i. Otherwise at least one of the components of $v - u$ is bigger in absolute value than $p/2^\mu$.

Now, suppose $\beta \not\equiv \alpha \pmod p$. Then

$$\Pr[\| v - u \| > p/2^\mu] \geq \Pr[\exists i \ :\ \mathrm{dist}_p(t_i\beta, a_i) > p/2^\mu] \geq$$
$$\Pr[\exists i \ :\ \mathrm{dist}_p(t_i\beta, t_i\alpha) > 2p/2^\mu] = 1 - (1-A)^d \geq 1 - \left(\frac{5}{2^\mu}\right)^d$$

Since $\beta \not\equiv \alpha \pmod p$ there are exactly $p-1$ values of $\beta \bmod p$ to consider. Hence, the probability there exists a lattice point contradicting the statement of the theorem is at most

$$(p-1)\cdot\left(\frac{5}{2^\mu}\right)^d < \frac{1}{2}$$

The last inequality follows from the fact that $d(\mu - \log_2 5) > \log p + 1$. This completes the proof of the theorem. ∎

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  - $\left|\left\{t \in (\mathbb{Z}/p\mathbb{Z})^\times : \mathrm{dist}_p(\beta t, \alpha t) \leq \frac{2p}{2^\mu}\right\}\right|$
    $= \left\lfloor p - \frac{2p}{2^\mu} \right\rfloor - \left\lceil \frac{2p}{2^\mu} \right\rceil$
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  - First, $\mathrm{MSB}_k(x)$ is the unique $t \in \mathbb{Z}$ such that $(t-1) \cdot \frac{p}{2^k} \leq x < t \cdot \frac{p}{2^k}$.

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  - Work in Isabelle locale fixing MSB_k operator and assuming $|x - \mathrm{MSB}_k(x)| < \frac{p}{2^k}$.

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```
definition game :: "((nat × nat) list ⇒ nat) ⇒ bool pmf" where
  "game 𝒜' = do {
    ts ← replicate_pmf d (pmf_of_set {1..<p});
    return_pmf (α = 𝒜' (map (λt. (t, 𝒪 t)) ts))
  }"
```

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