A Verified Cost Model for Call-by-Push-Value

ITP2025

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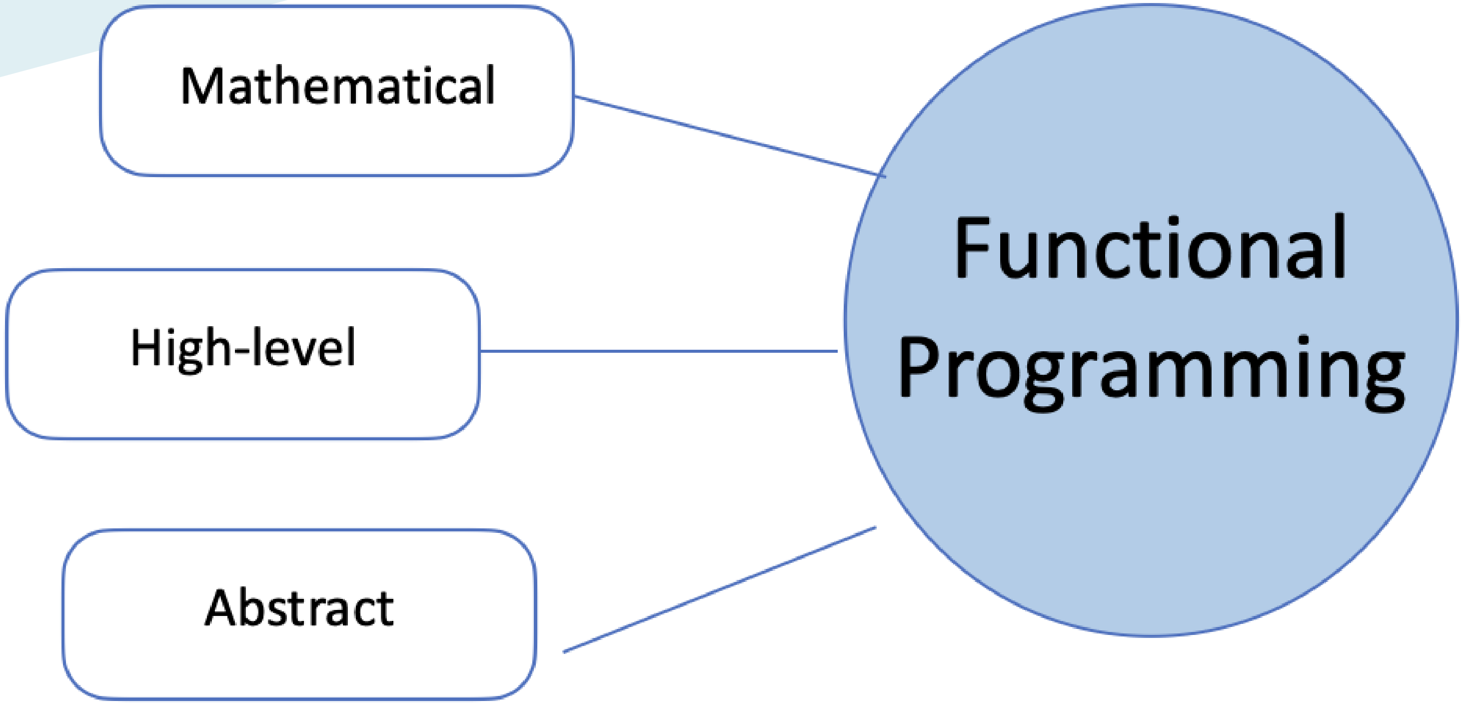
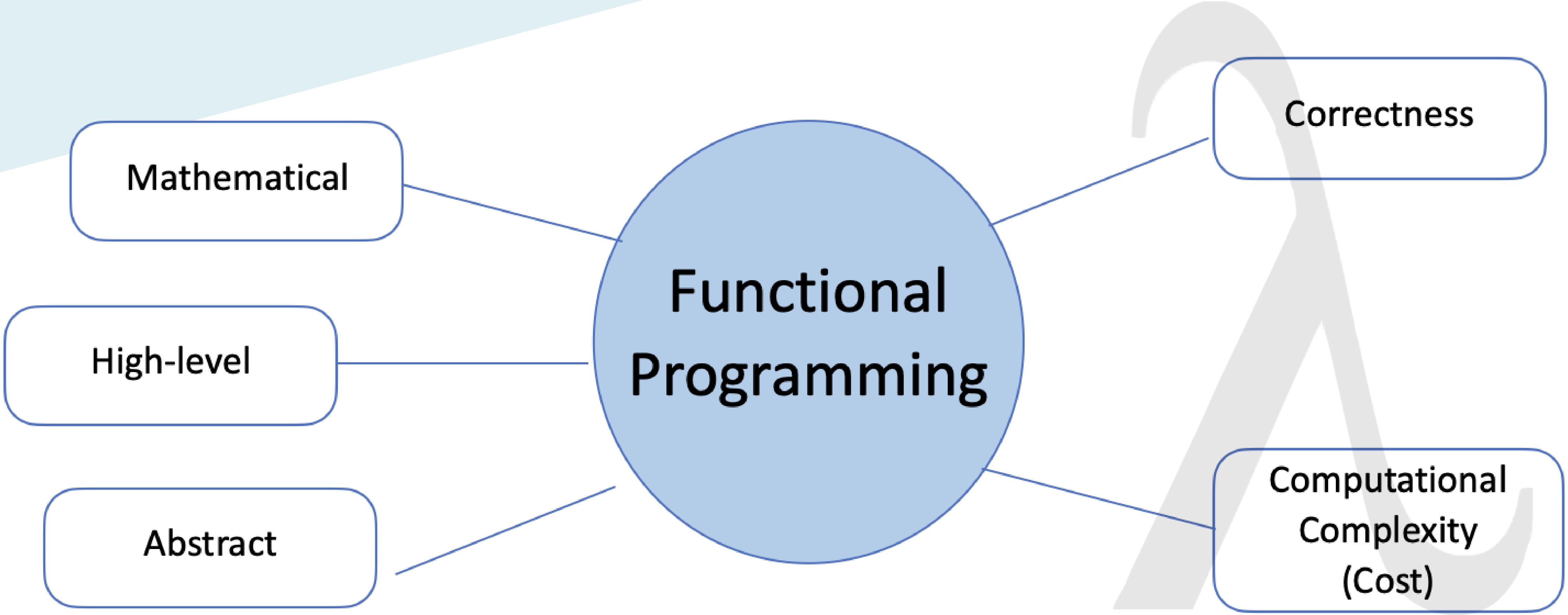
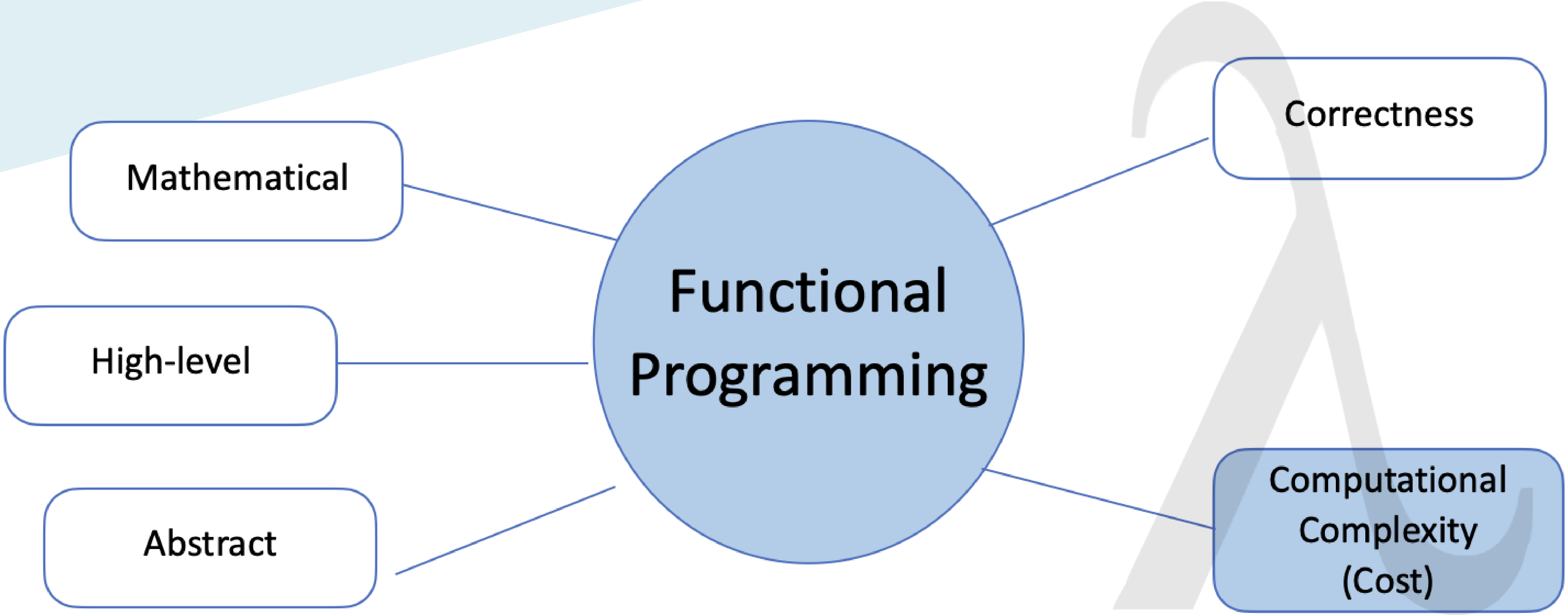
Problem and Motivation



Problem and Motivation



Problem and Motivation



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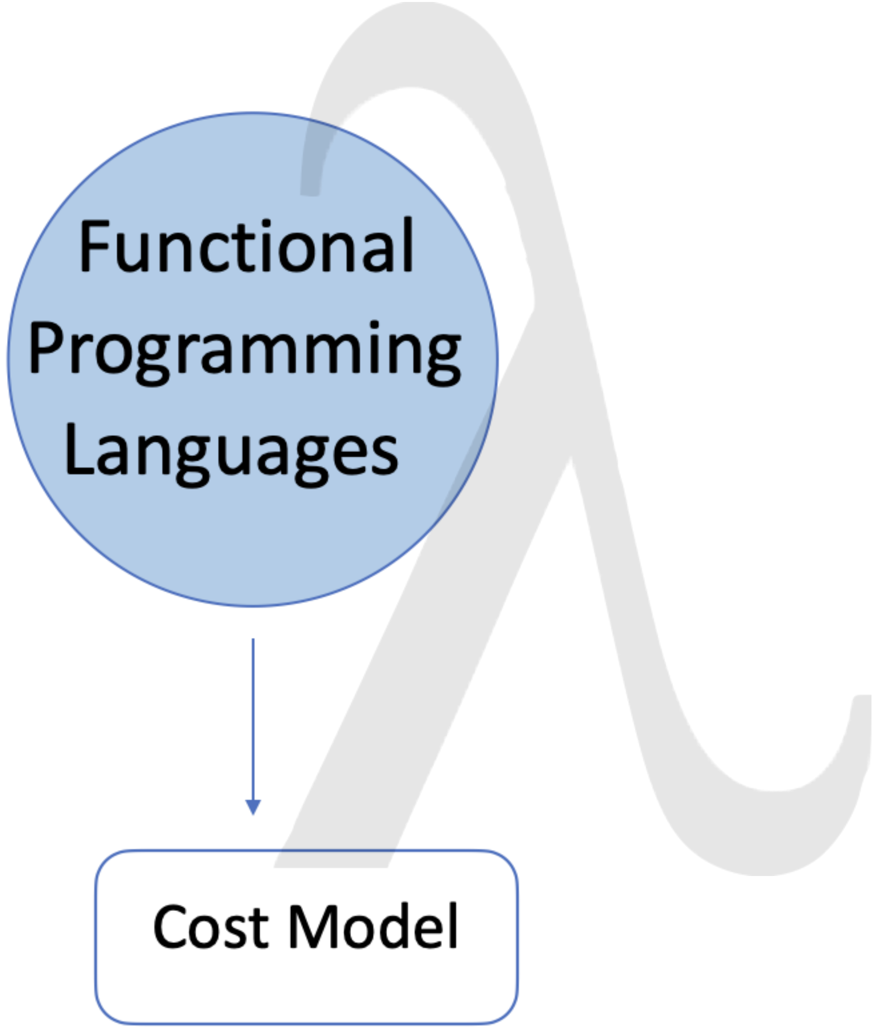

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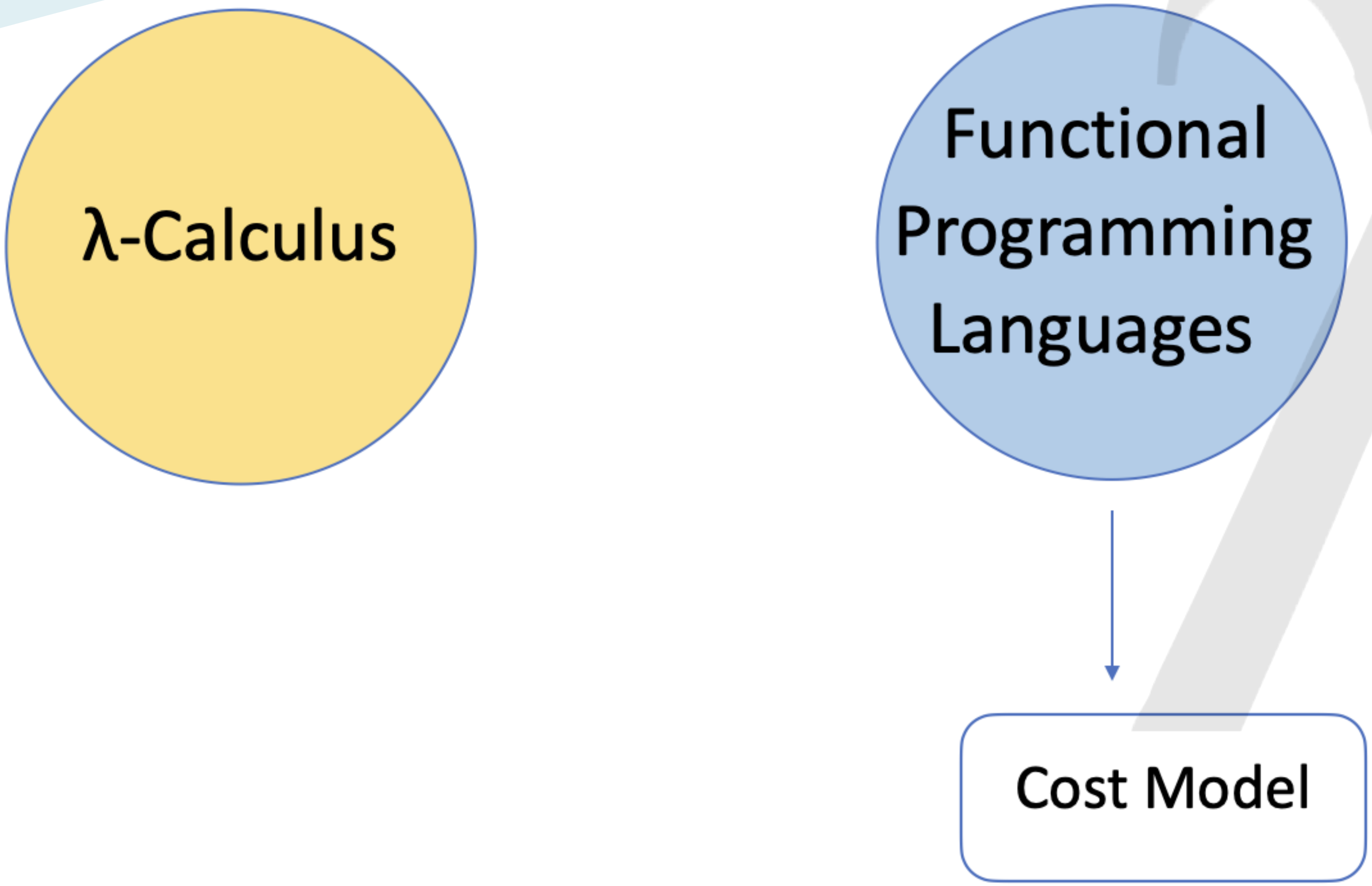

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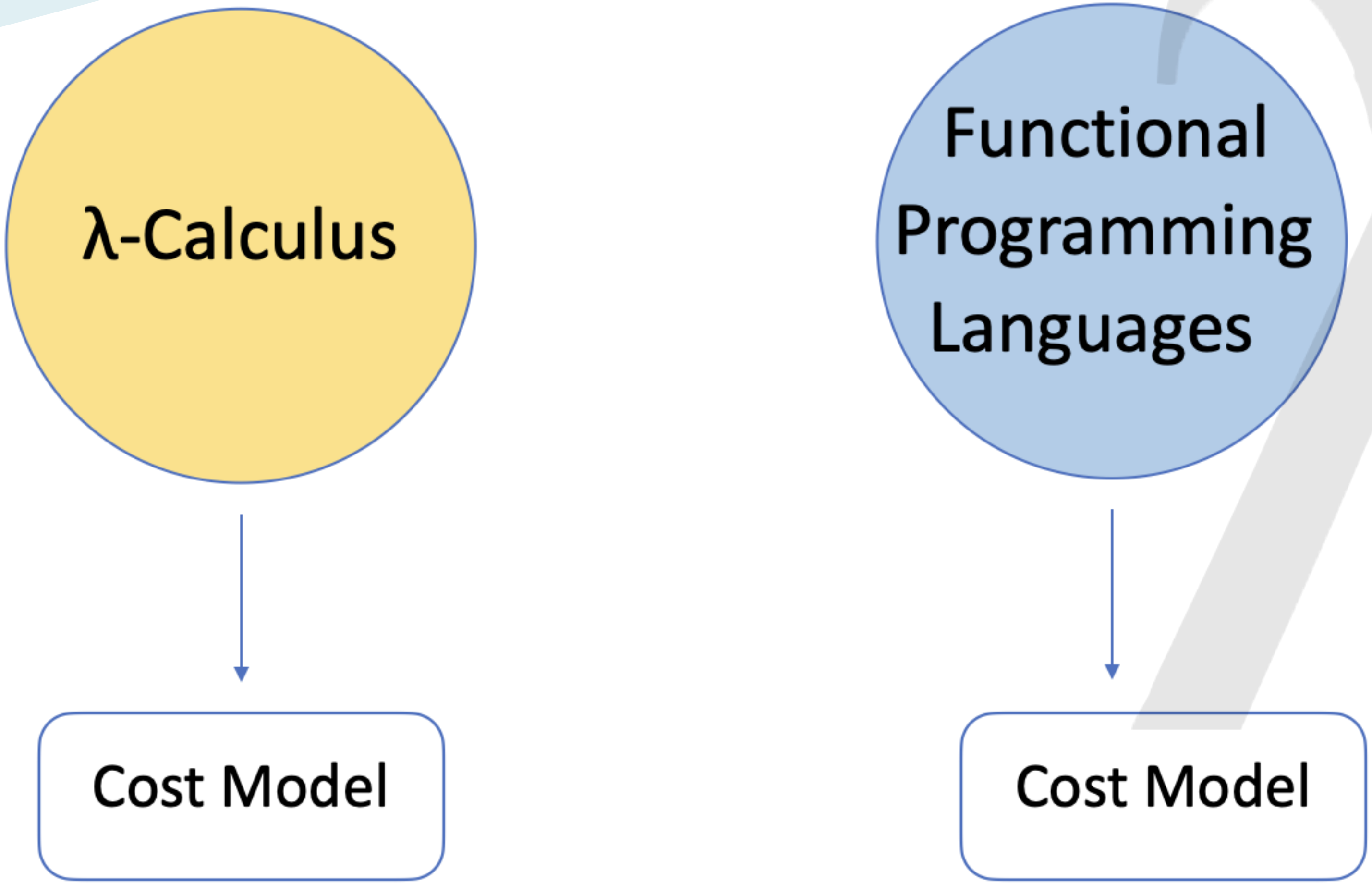
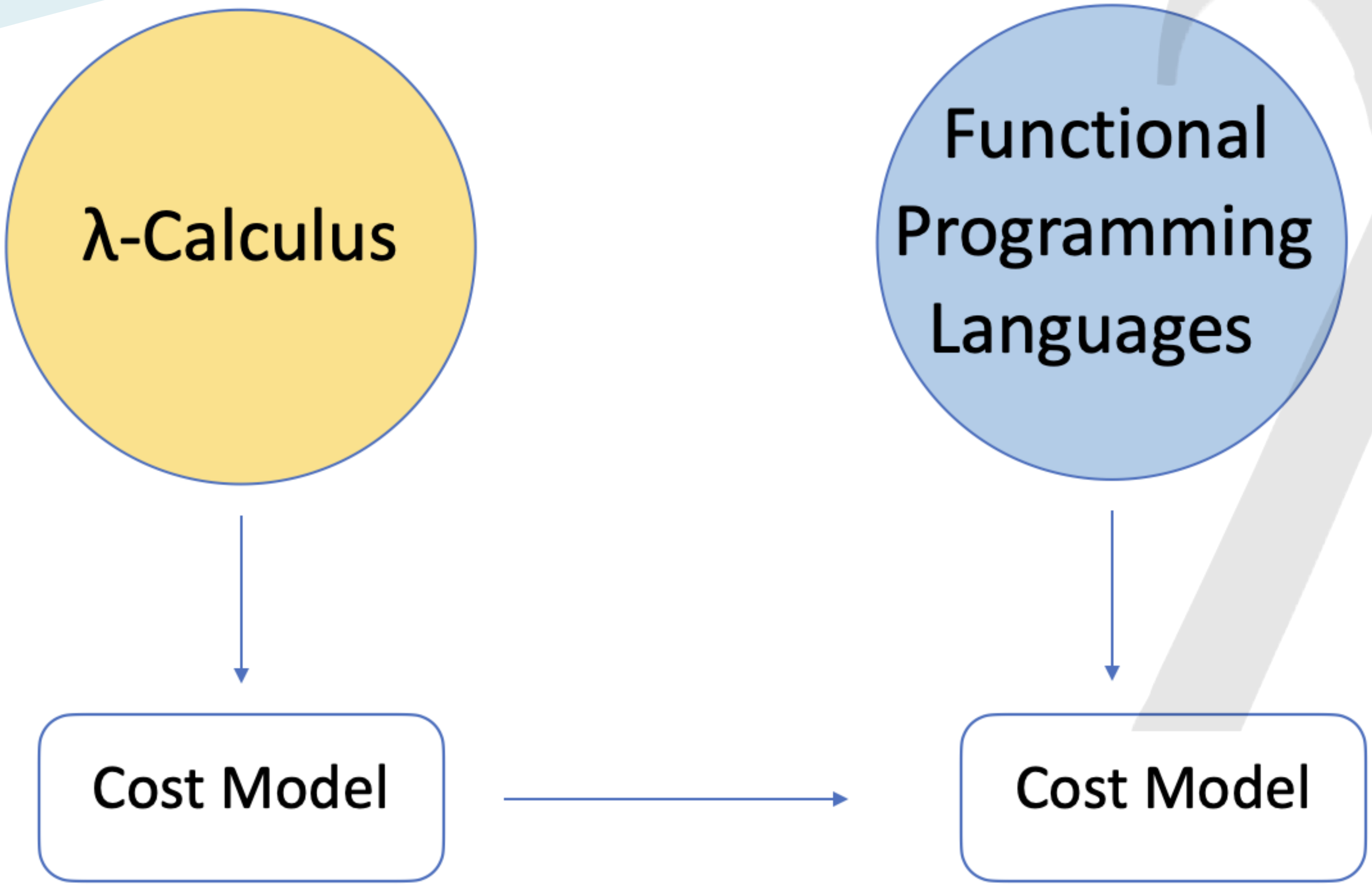


Problem and Motivation



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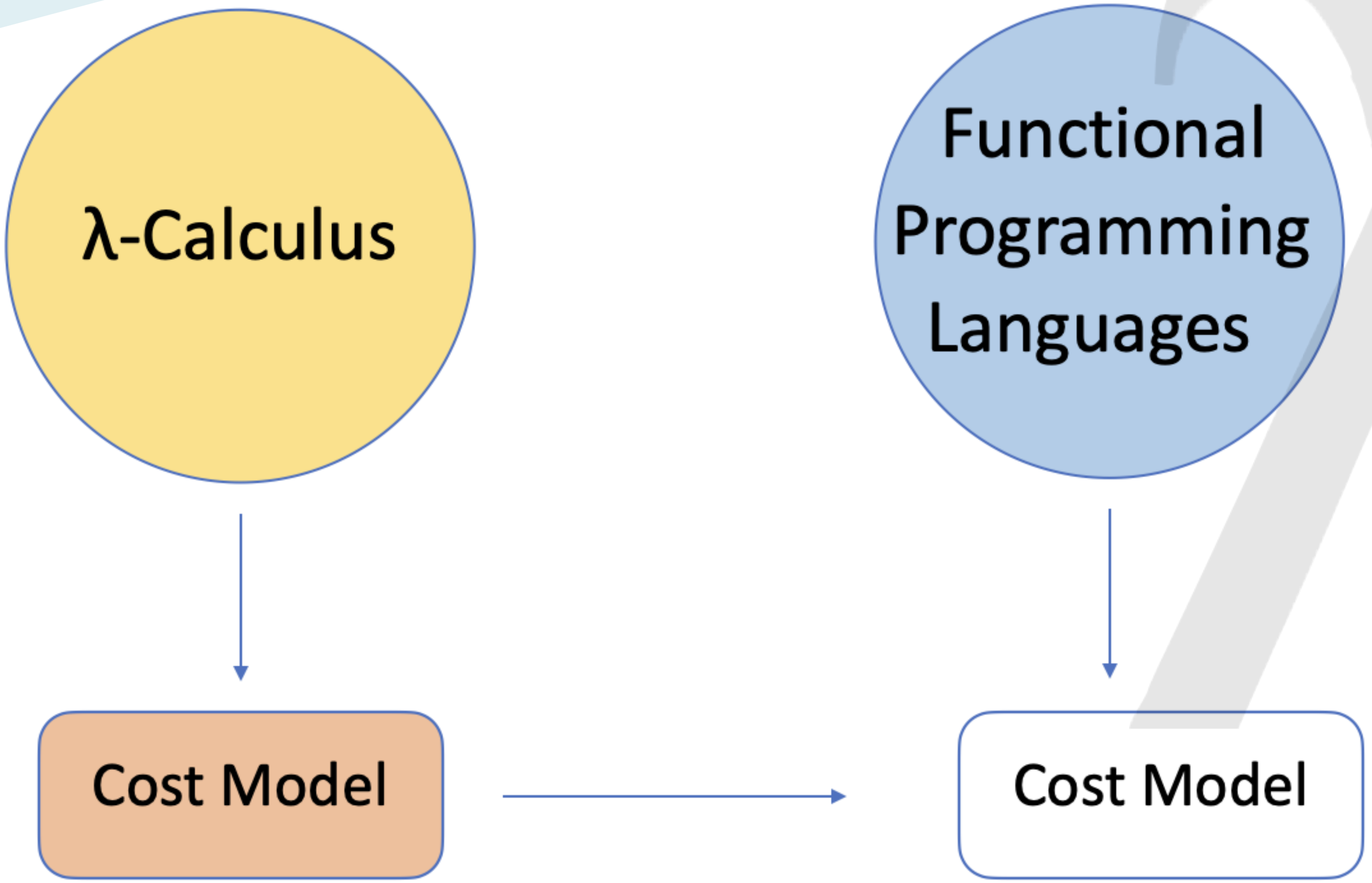

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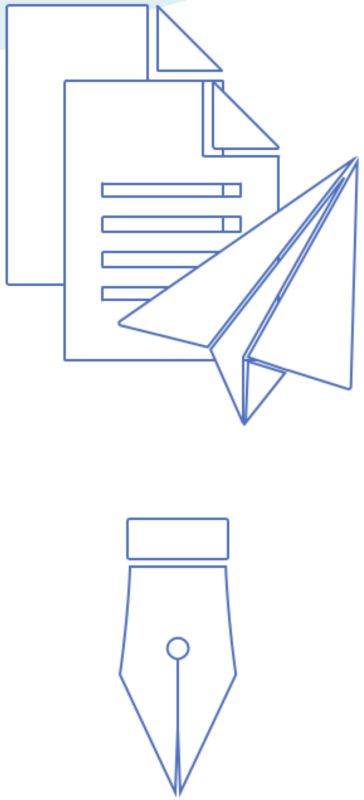
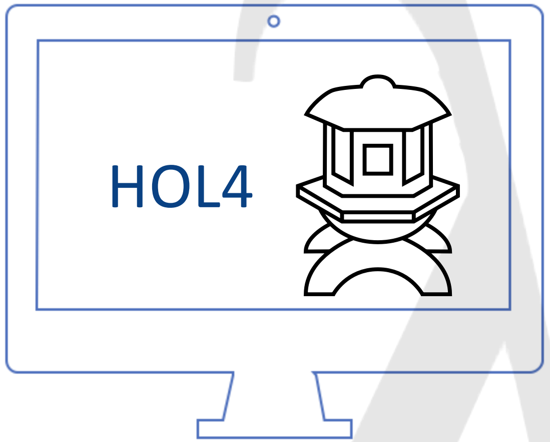
Formal Verification





Interactive Theorem Prover

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Background – Lambda Calculus

λ-expression $s, t := x \mid (\lambda x.\, s) \mid (s\, t)$

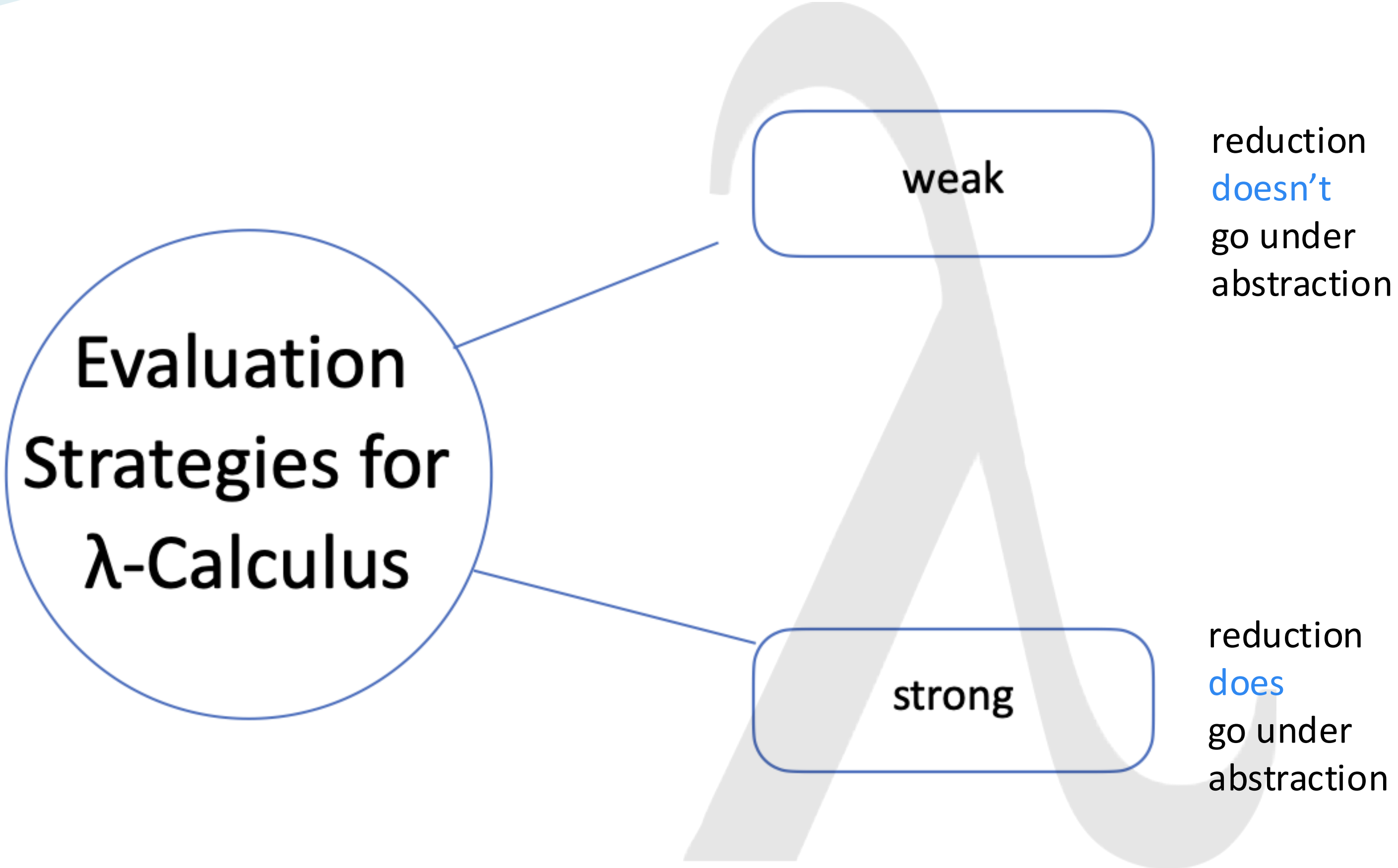
(β-reduction). $(\lambda x.s)\, t \to_\beta s[x := t]$

(Substitution). $s[x := t]$

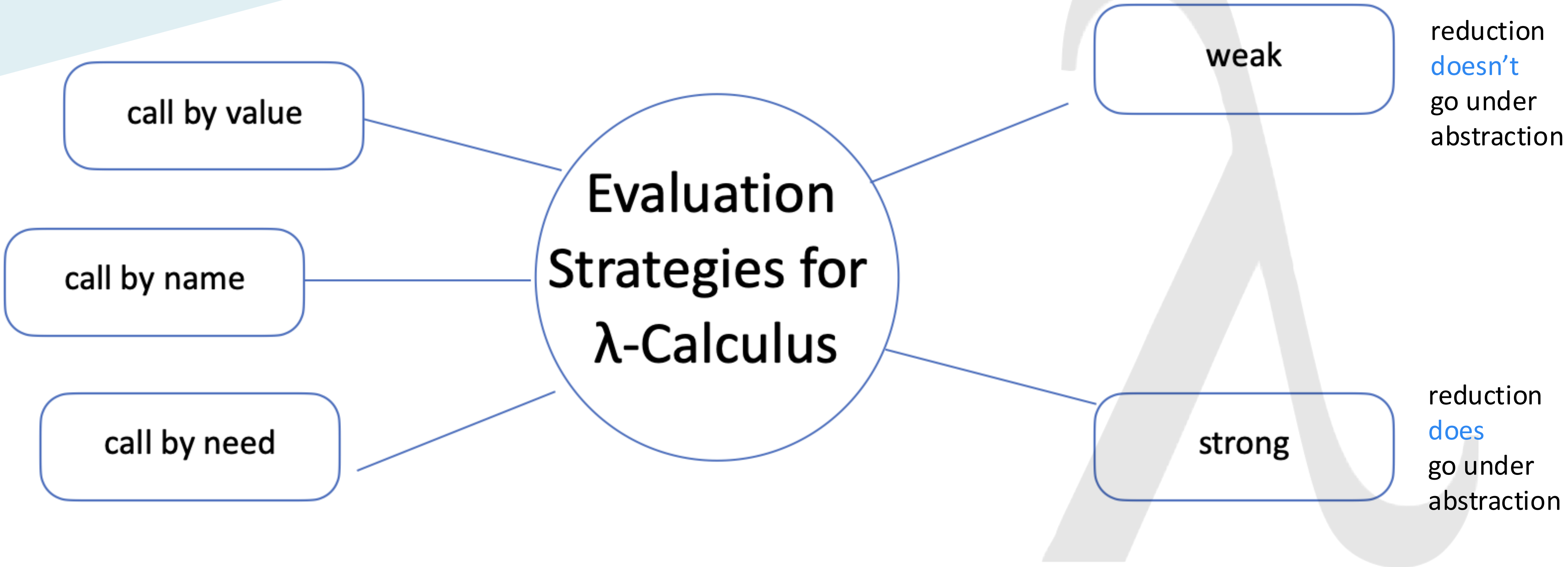
Background – Lambda Calculus

Evaluation Strategies for λ-Calculus

Background – Lambda Calculus



Background – Lambda Calculus



Background – Lambda Calculus

Evaluates the function argument before passing them into the function call

call by value

call by name

call by need

Evaluation Strategies for λ-Calculus

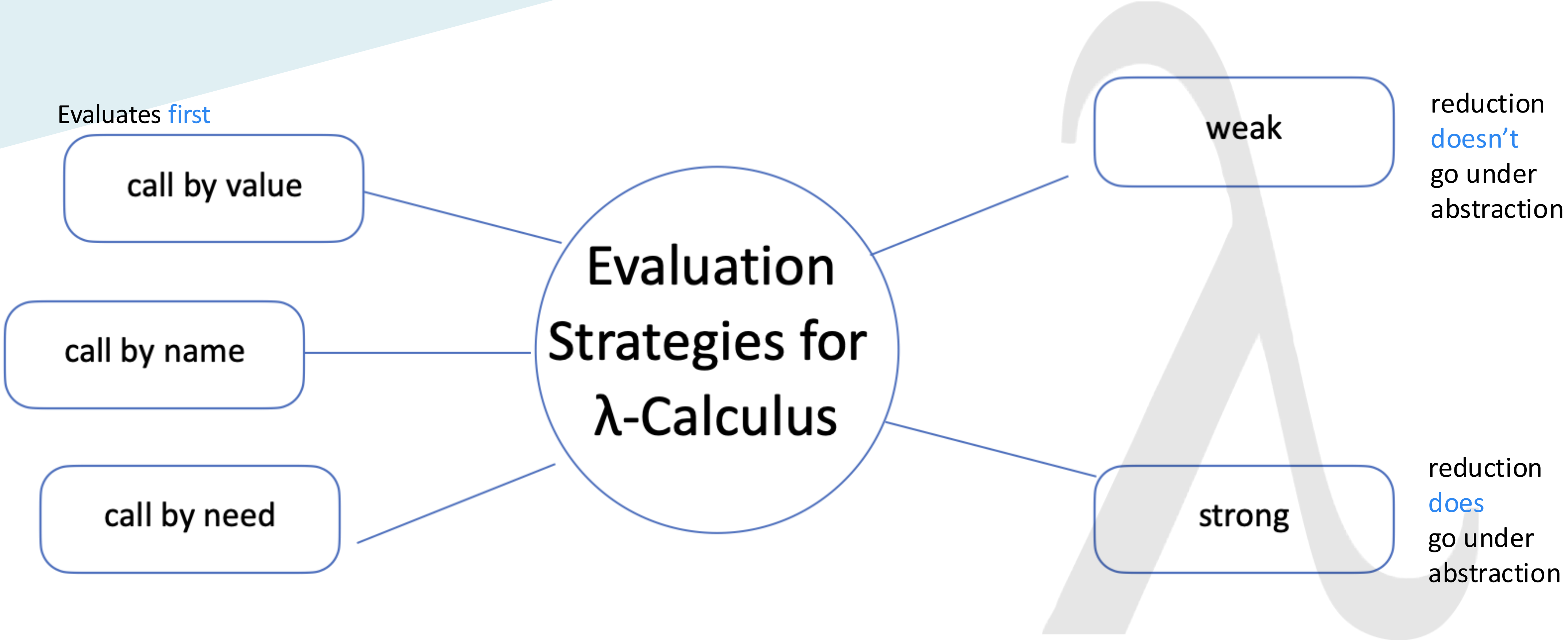
weak

reduction doesn't go under abstraction

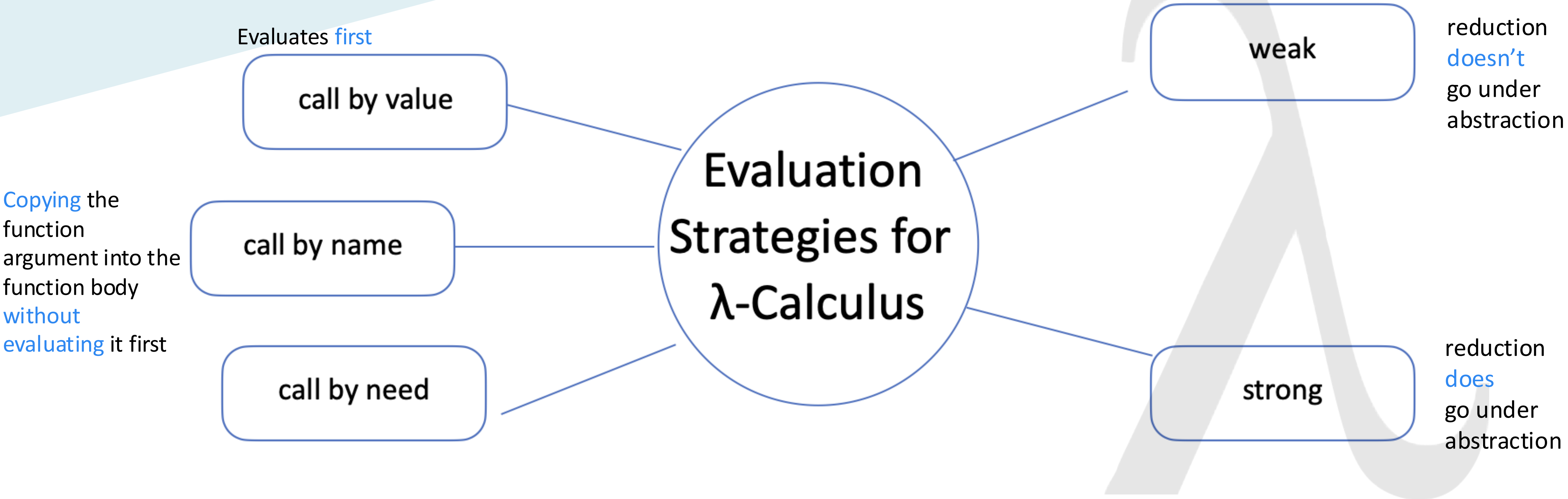
strong

reduction does go under abstraction

Background – Lambda Calculus



Background – Lambda Calculus



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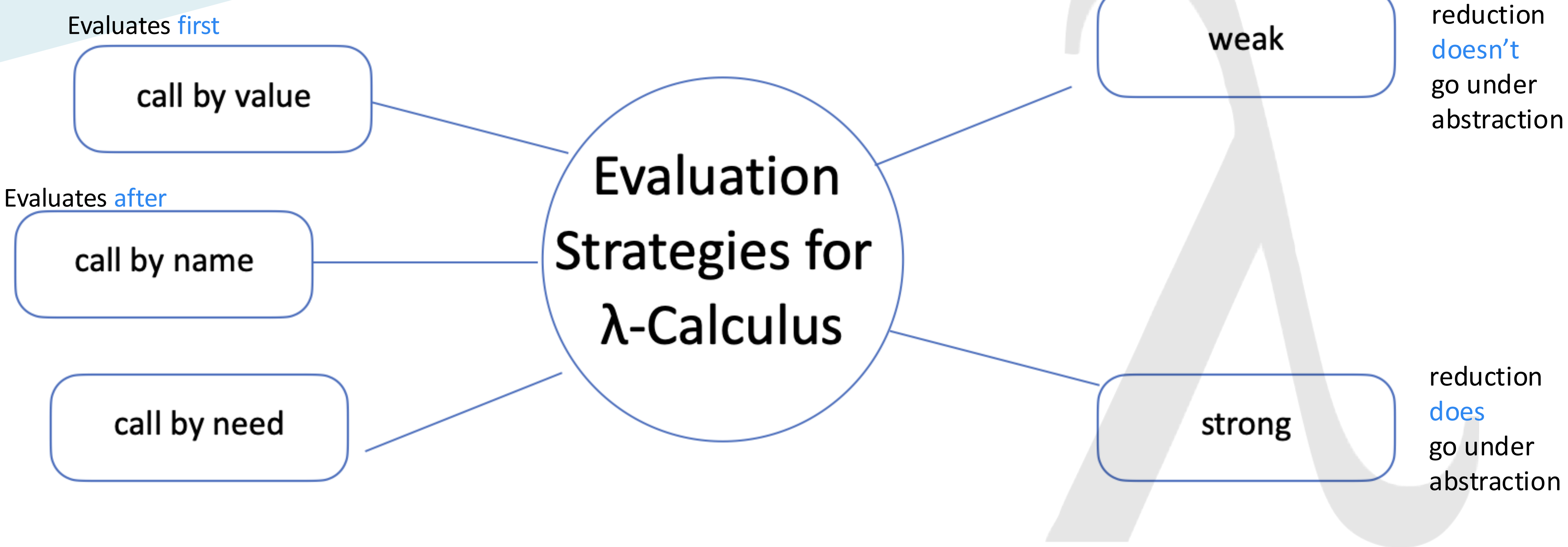
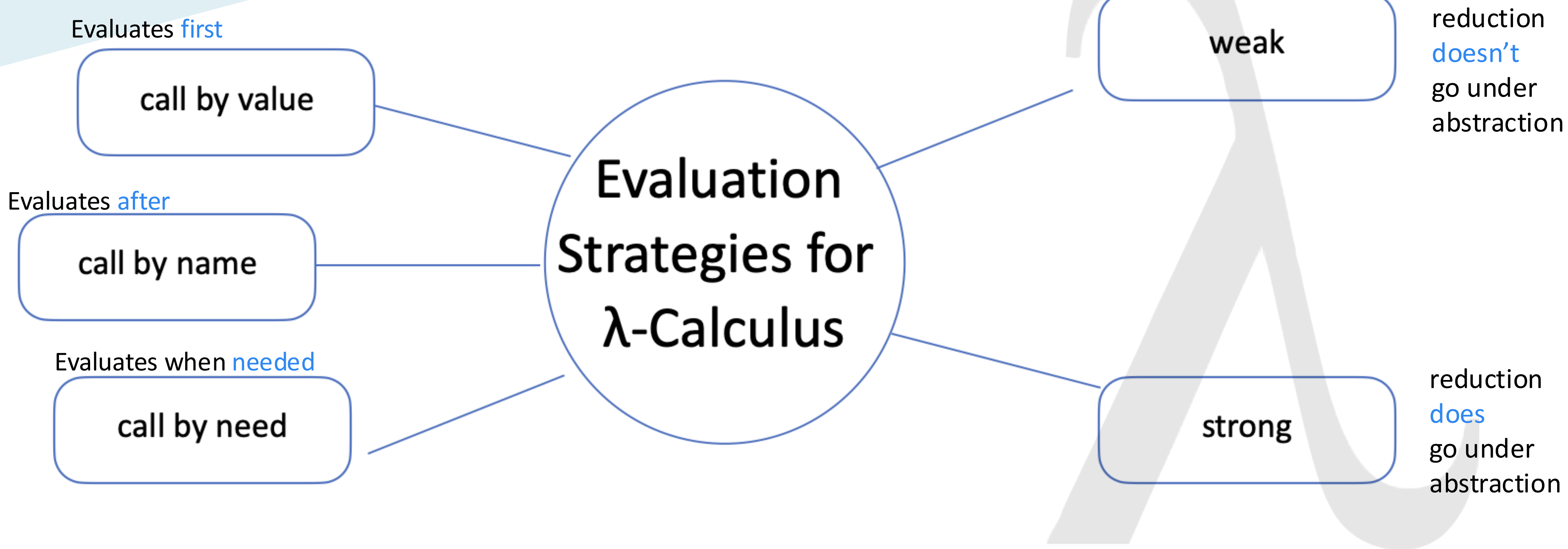
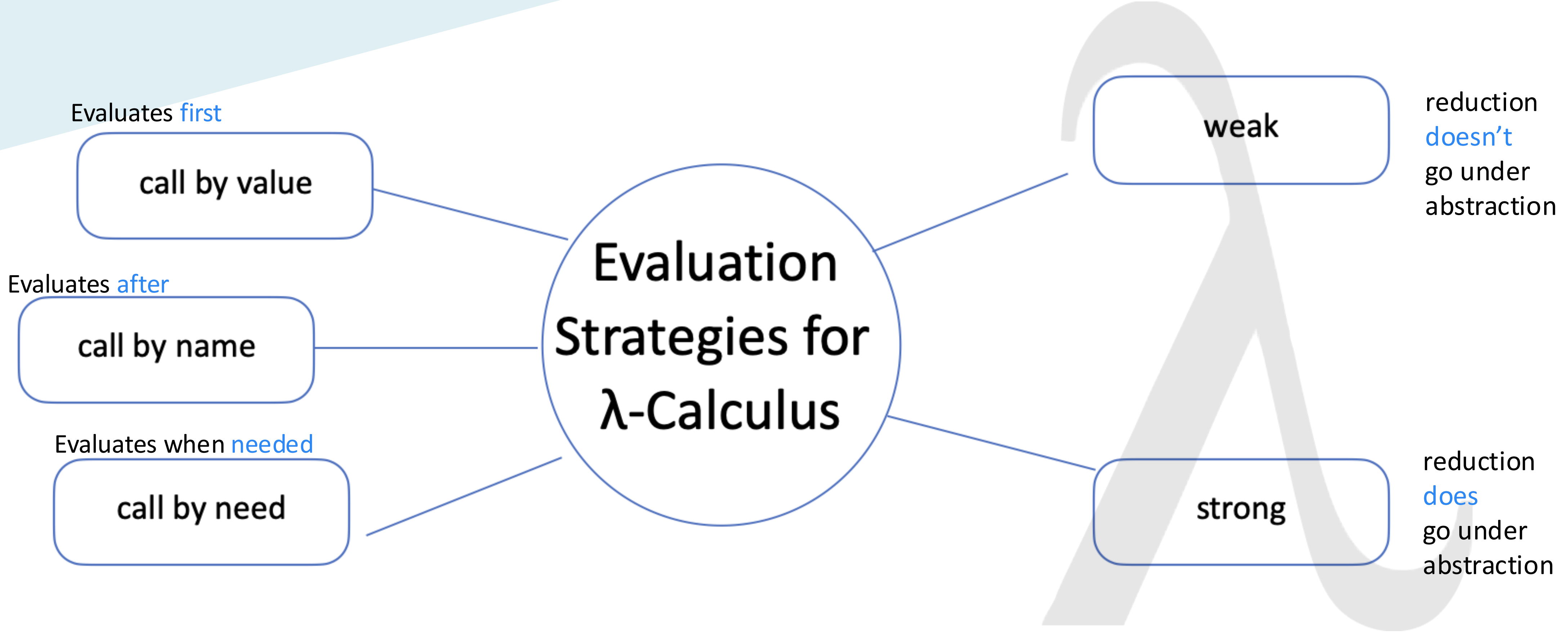


Background – Lambda Calculus



Background – Lambda Calculus



Need different cost models!

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A Verified Cost Model for Call-by-Push-Value

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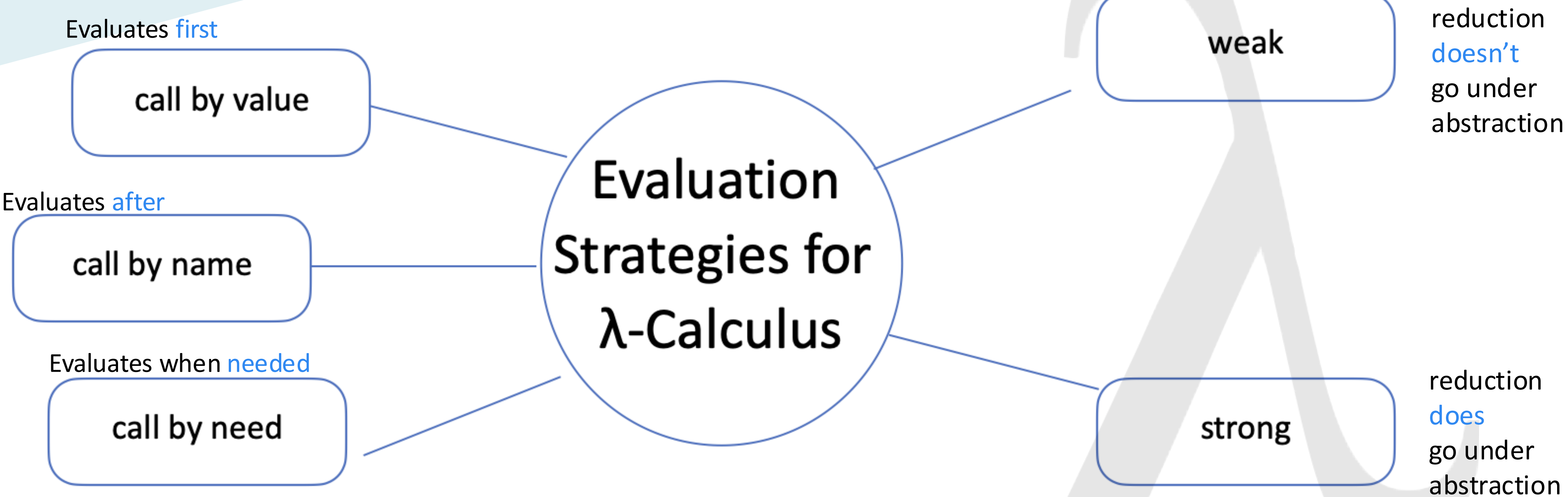
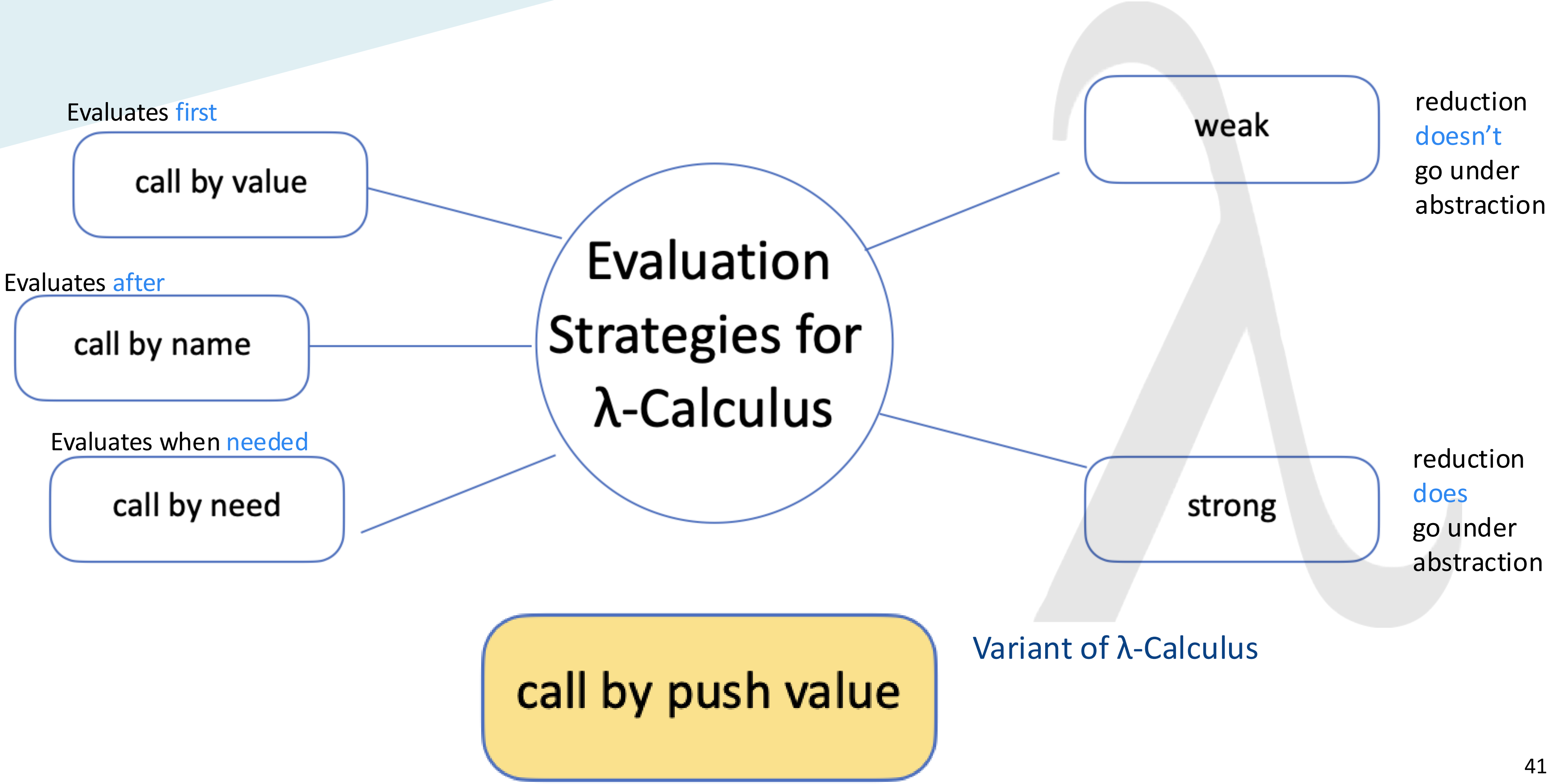
Background – CBPV [Levy, 1999]

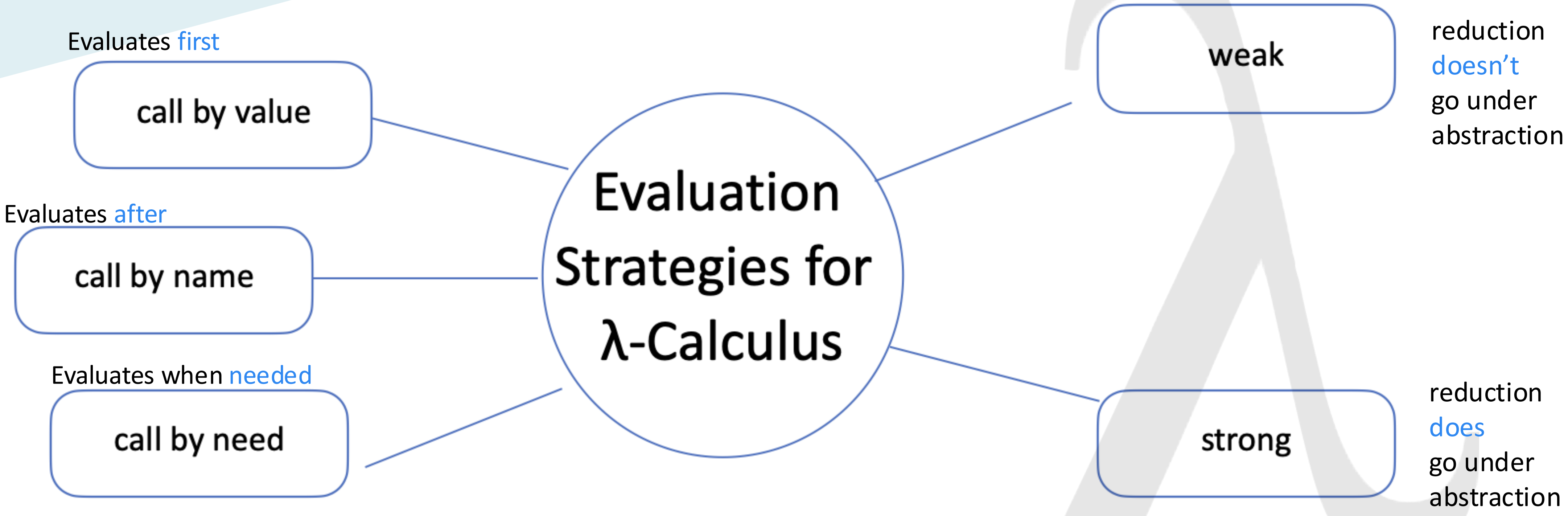


call by push value

Background – CBPV



Background – CBPV

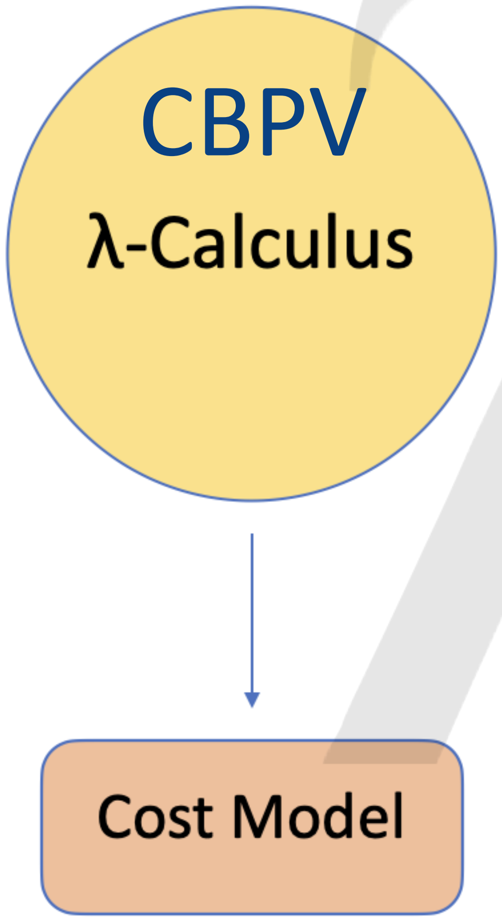


call by push value

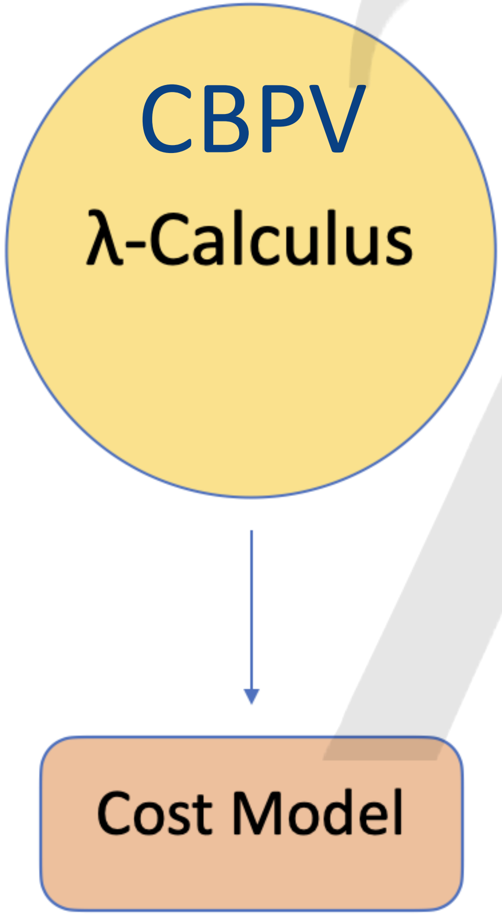
Variant of λ-Calculus

uses the pair "thunk" and "force" to let the programmer freely decide when they wish to evaluate which evaluation

A Verified Cost Model for CBPV

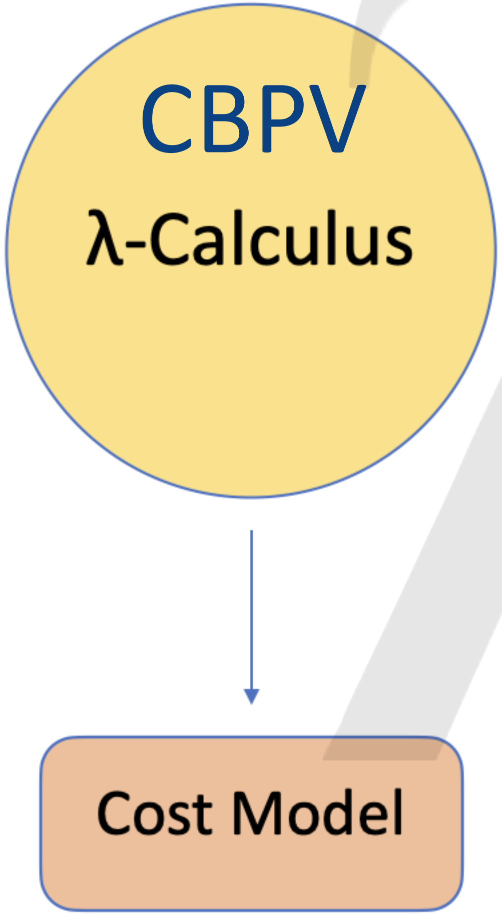


A Verified Cost Model for CBPV



<u>Time</u>: Number of β-reductions
<u>Space</u>: Size of the biggest intermediate term

A Verified Cost Model for CBPV



?

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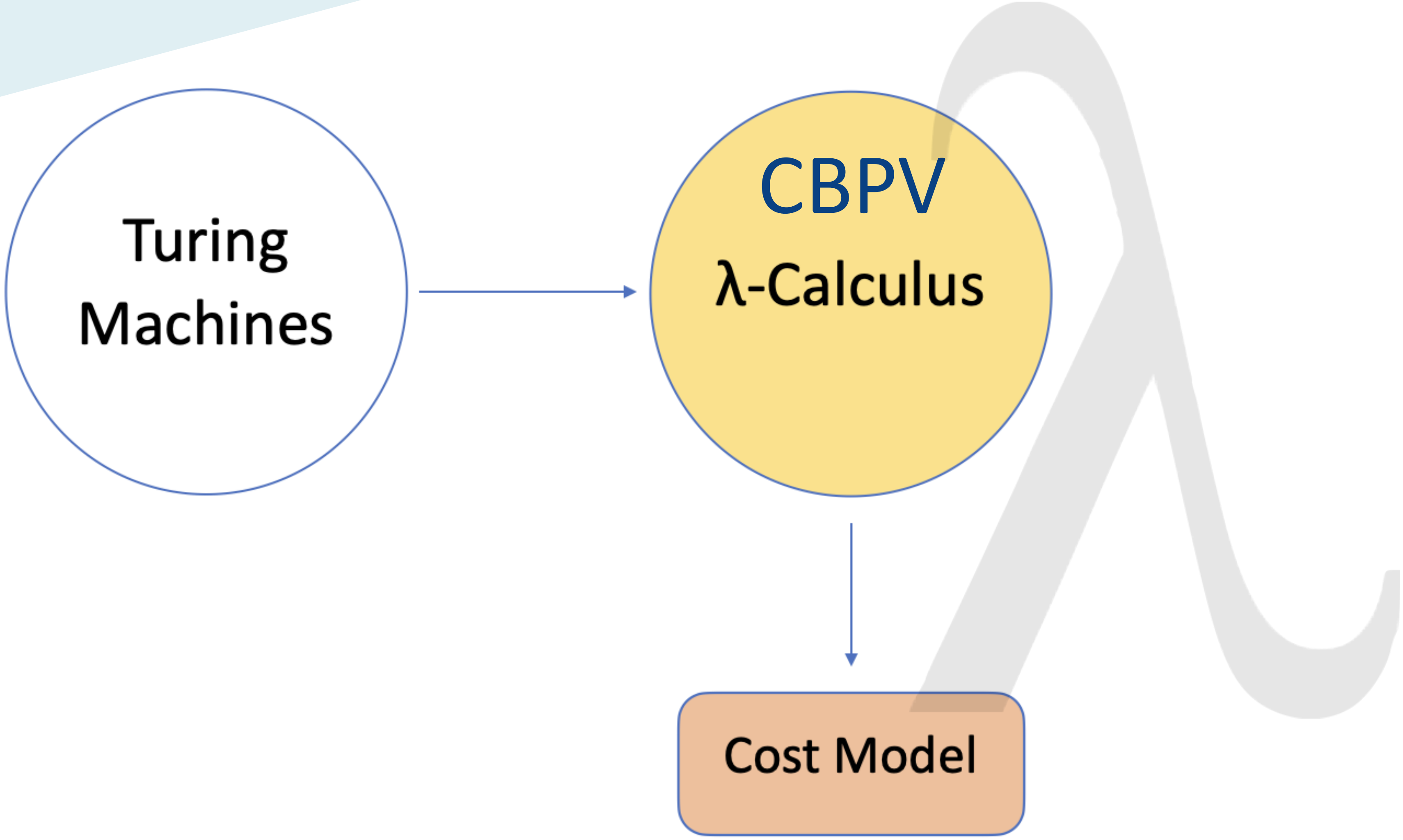
Space: Size of the biggest intermediate term

A Verified Cost Model for CBPV



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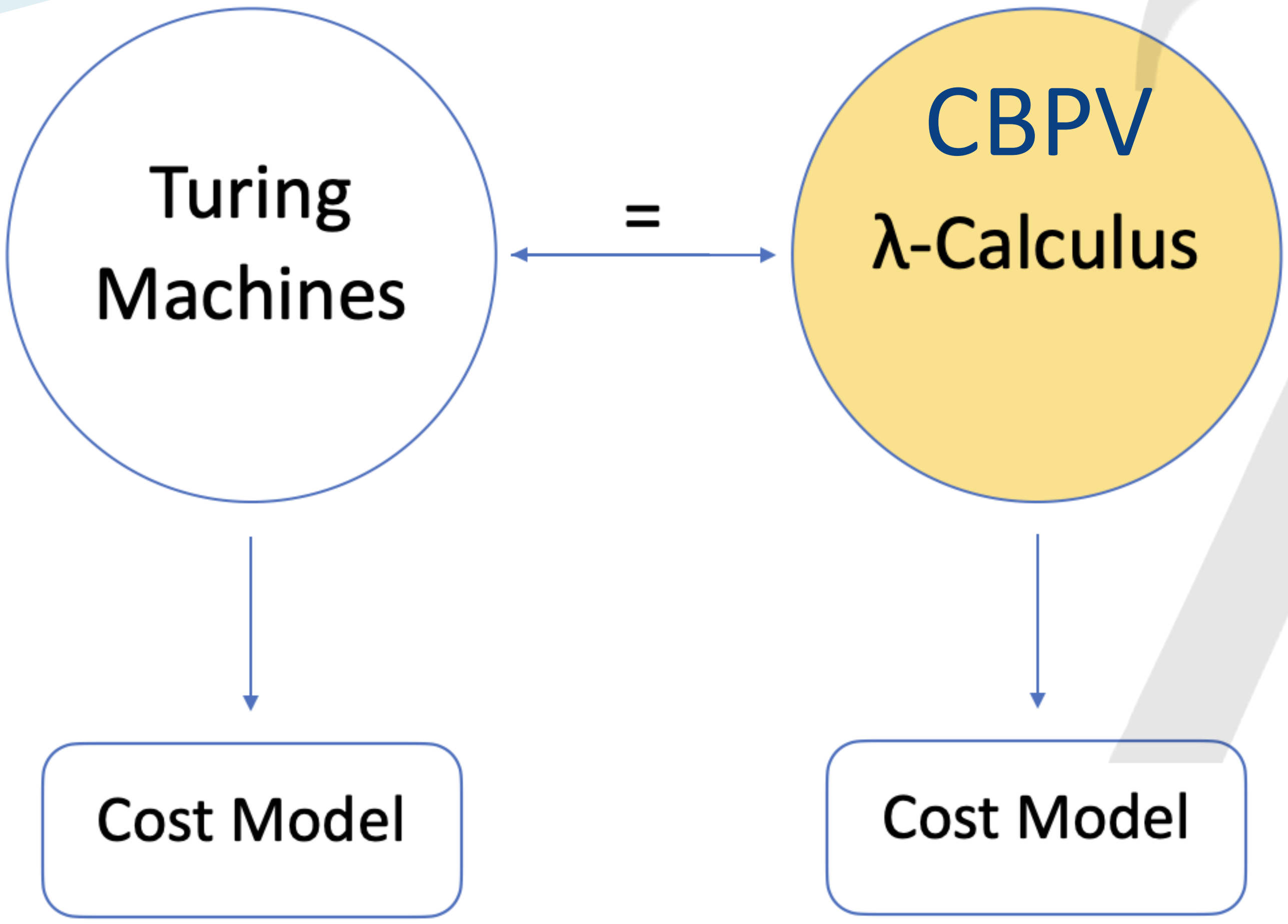

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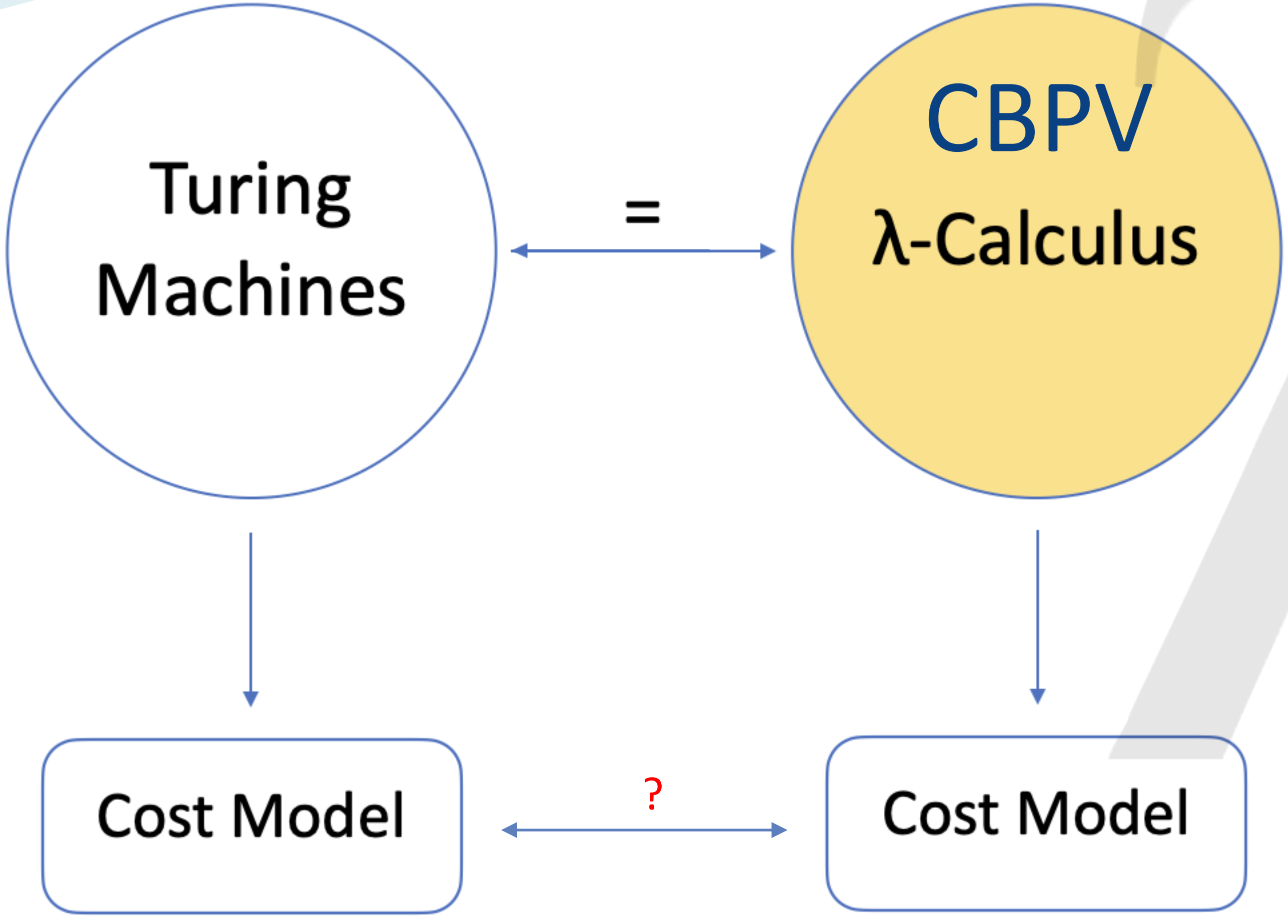

The Invariance Thesis

Slot and van Emde Boas (1984):

“*Reasonable* machines simulate each other with *polynomially* bounded overhead in *time* and *constant* factor overhead in *space*”

The Invariance Thesis

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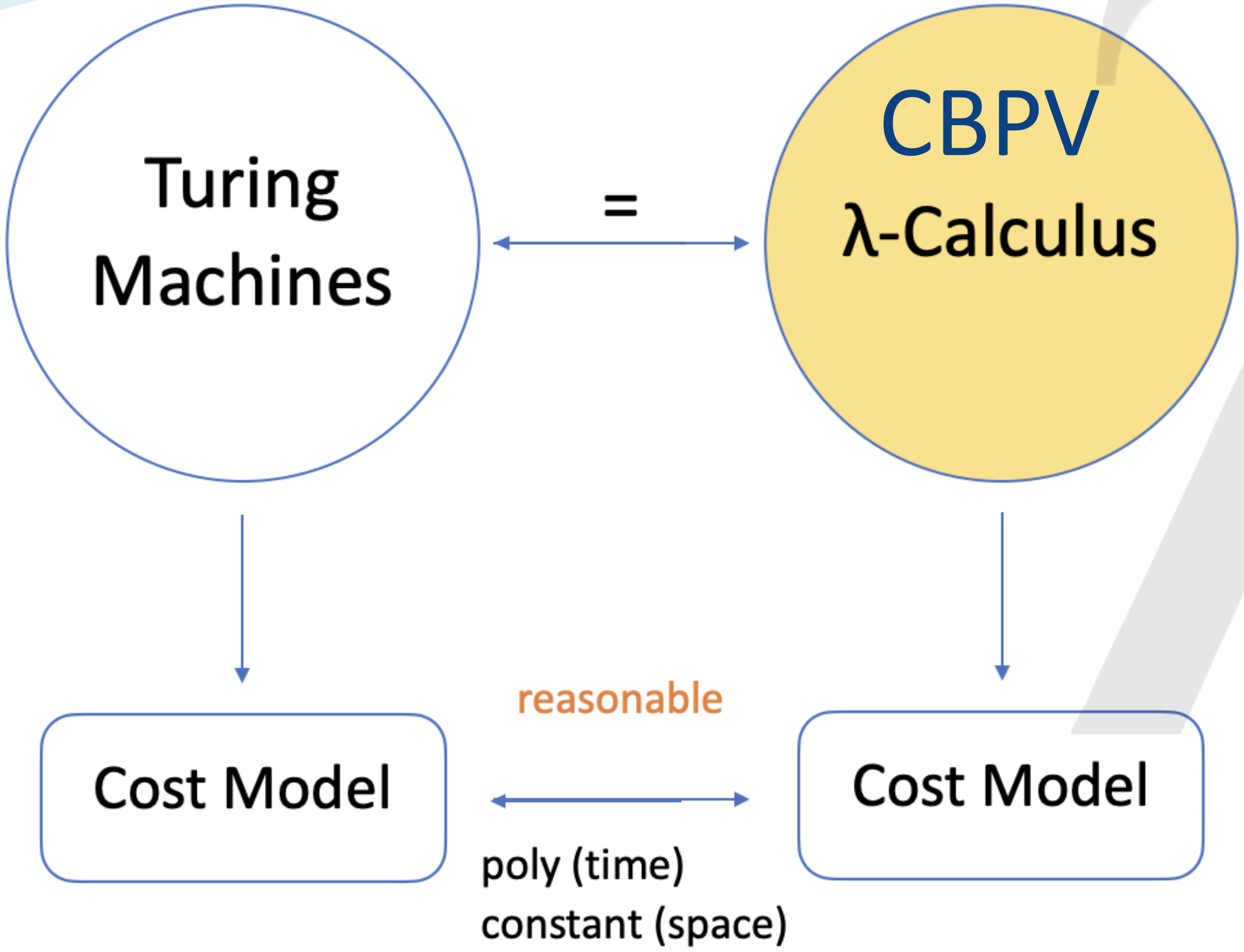
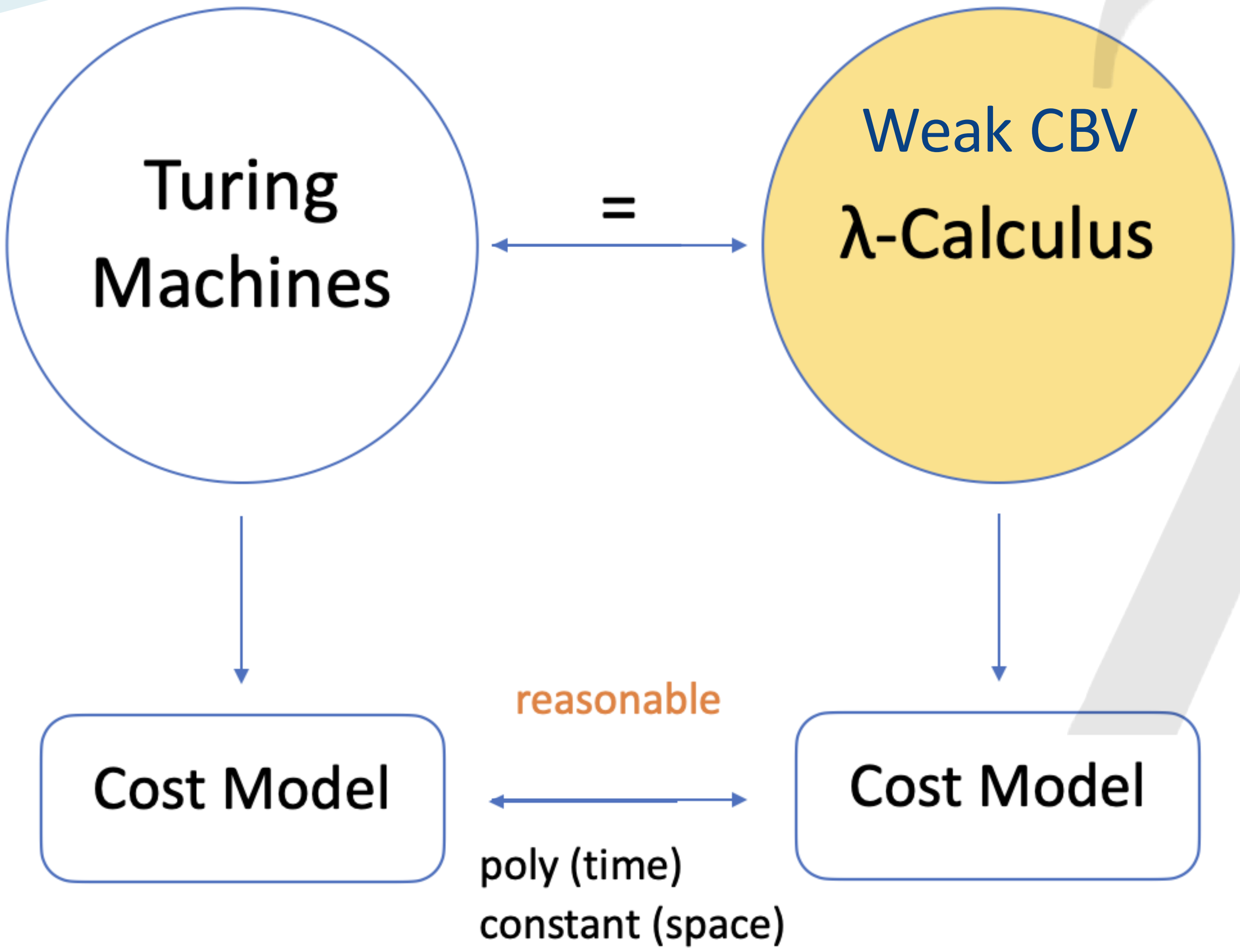
"*Reasonable* machines simulate each other with *polynomially* bounded overhead in *time* and *constant* factor overhead in *space*"

Turing machines are reasonable

A Verified Cost Model for CBPV



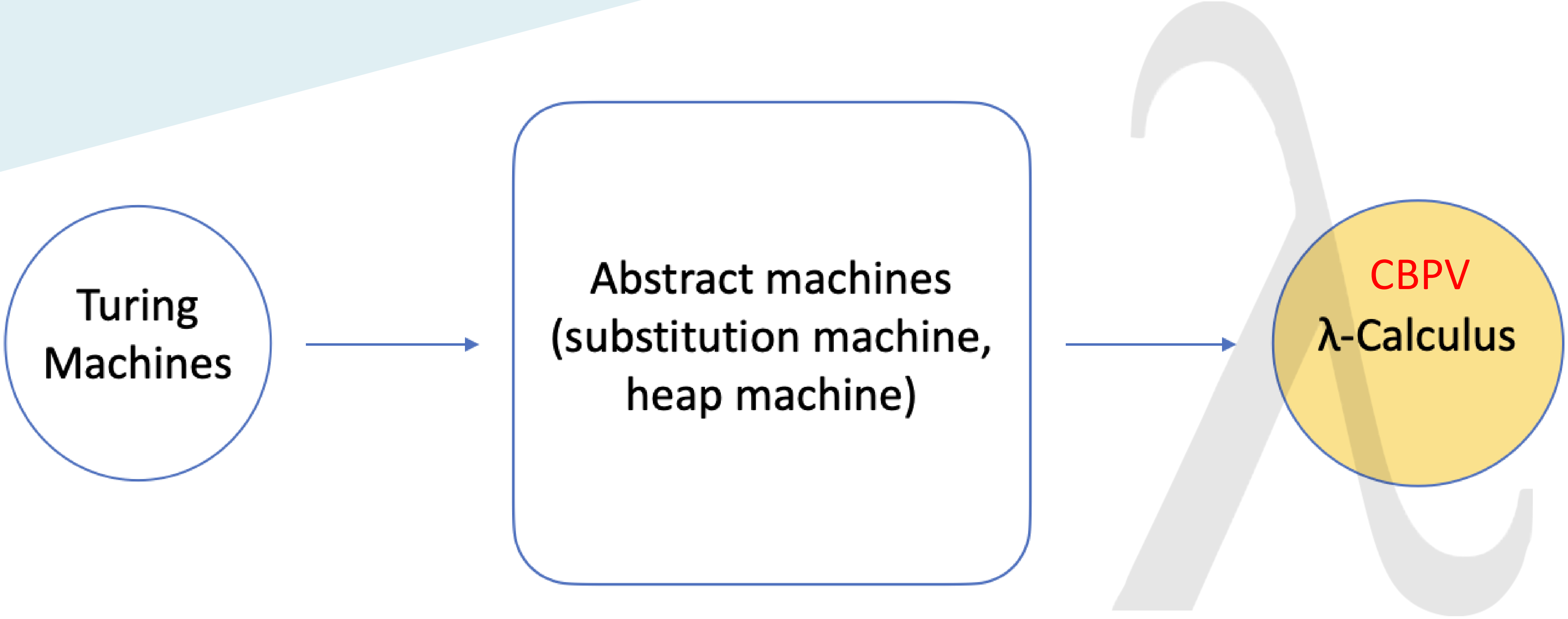
Related Work [Forster, Kunze, and Roth 2020]



Related Work [Forster, Kunze, and Roth 2020]

Turing Machines → Abstract machines (substitution machine, heap machine) → Weak CBV λ-Calculus

A Verified Cost Model for CBPV



A Verified Cost Model for CBPV

Turing Machines → Abstract machines (substitution machine, heap machine) → CBPV λ-Calculus

CBPV

de Bruijn syntax

Definition 33 (Syntax for CBPV).

$$
\begin{array}{llllll}
\textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & \\
\textit{Computations} & \mathtt{M} & := & \lambda.\,\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid \quad \mathsf{ret}\ \mathtt{V} \mid \\
 & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M}
\end{array}
$$

CBPV

Definition 33 (Syntax for CBPV).

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CBPV

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 & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M}
\end{array}
$$

CBPV

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$$\begin{array}{llll} \textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid \mathsf{thunk}\ \mathtt{M} \\ \textit{Computations} & \mathtt{M} & := & \lambda.\,\mathtt{M} \mid \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid \mathsf{force}\ \mathtt{V} \mid \mathsf{ret}\ \mathtt{V} \mid \\ & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M} \end{array}$$

CBPV

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CBPV

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\end{array}
$$

pseq m2 m1 n = seq m2 (seq m1 n)

CBPV

Definition 34. *(Substitution function)*

$$
\begin{aligned}
(\lambda.\, m)^i_u &= \lambda.\, (m^{i+1}_u) \\
(\mathsf{app}\; m\; v)^i_u &= \mathsf{app}\; (m^i_u)\; (v^i_u) \\
(\mathsf{ret}\; v)^i_u &= \mathsf{ret}\; (v^i_u) \\
(\mathsf{seq}\; m\; n)^i_u &= \mathsf{seq}\; (m^i_u)\; (n^{i+1}_u) \\
(\mathsf{pseq}\; m_2\; m_1\; n)^i_u &= \mathsf{pseq}\; ({m_2}^i_u)\; ({m_1}^i_u)\; (n^{i+2}_u)
\end{aligned}
$$

$$
\begin{aligned}
(\mathsf{var}\; x)^i_u &= u \;(\mathit{if}\; x = i) \\
(\mathsf{var}\; x)^i_u &= \mathsf{var}\; x \;(\mathit{if}\; x \neq i) \\
(\mathsf{thunk}\; m)^i_u &= \mathsf{thunk}\; (m^i_u) \\
(\mathsf{force}\; v)^i_u &= \mathsf{force}\; (v^i_u) \\
(\mathsf{let}\; v.\, \mathsf{in}\; m)^i_u &= \mathsf{let}\; (v^i_u).\, \mathsf{in}\; (m^{i+1}_u)
\end{aligned}
$$

CBPV

$$
\frac{}{\lambda.\, m \Downarrow \lambda.\, m}
\qquad
\frac{}{\mathsf{ret}\; v \Downarrow \mathsf{ret}\; v}
\qquad
\frac{m \Downarrow m'}{\mathsf{force}\,(\mathsf{thunk}\; m) \Downarrow m'}
\qquad
\frac{m_v^0 \Downarrow m'}{(\mathsf{let}\; v.\, \mathsf{in}\; m) \Downarrow m'}
$$

$$
\frac{m \Downarrow \lambda.\, n \qquad n_v^0 \Downarrow n'}{\mathsf{app}\; m\; v \Downarrow n'}
\qquad
\frac{m \Downarrow \mathsf{ret}\; v \qquad n_v^0 \Downarrow n'}{\mathsf{seq}\; m\; n \Downarrow n'}
$$

$$
\frac{m_1 \Downarrow \mathsf{ret}\; v_1 \qquad m_2 \Downarrow \mathsf{ret}\; v_2 \qquad (n_{v_1}^0)_{v_2}^1 \Downarrow n'}{\mathsf{pseq}\; m_2\; m_1\; n \Downarrow n'}
$$

FIGURE 4.3: Big-Step Semantics for CBPV

CBPV

$$\frac{}{\lambda.\, m \Downarrow \lambda.\, m} \qquad \frac{}{\mathsf{ret}\, v \Downarrow \mathsf{ret}\, v} \qquad \frac{m \Downarrow m'}{\mathsf{force}\,(\mathsf{thunk}\, m) \Downarrow m'} \qquad \frac{m^0_v \Downarrow m'}{(\mathsf{let}\, v.\, \mathsf{in}\, m) \Downarrow m'}$$

$$\frac{m \Downarrow \lambda.\, n \qquad n^0_v \Downarrow n'}{\mathsf{app}\, m\, v \Downarrow n'} \qquad \frac{m \Downarrow \mathsf{ret}\, v \qquad n^0_v \Downarrow n'}{\mathsf{seq}\, m\, n \Downarrow n'}$$

$$\frac{m_1 \Downarrow \mathsf{ret}\, v_1 \qquad m_2 \Downarrow \mathsf{ret}\, v_2 \qquad (n^0_{v_1})^1_{v_2} \Downarrow n'}{\mathsf{pseq}\, m_2\, m_1\, n \Downarrow n'}$$

FIGURE 4.3: Big-Step Semantics for CBPV

CBPV

$$\frac{}{\lambda.\, m \Downarrow \lambda.\, m} \qquad \frac{}{\mathsf{ret}\, v \Downarrow \mathsf{ret}\, v} \qquad \frac{m \Downarrow m'}{\mathsf{force}\,(\mathsf{thunk}\, m) \Downarrow m'} \qquad \frac{m_v^0 \Downarrow m'}{(\mathsf{let}\, v.\, \mathsf{in}\, m) \Downarrow m'}$$

$$\frac{m \Downarrow \lambda.\, n \qquad n_v^0 \Downarrow n'}{\mathsf{app}\, m\, v \Downarrow n'} \qquad \frac{m \Downarrow \mathsf{ret}\, v \qquad n_v^0 \Downarrow n'}{\mathsf{seq}\, m\, n \Downarrow n'}$$

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FIGURE 4.3: Big-Step Semantics for CBPV

CBPV

$$\frac{}{\lambda.\, m \Downarrow_0 \lambda.\, m} \qquad \frac{}{\mathsf{ret}\; v \Downarrow_0 \mathsf{ret}\; v} \qquad \frac{m \Downarrow_k m'}{\mathsf{force}\;(\mathsf{thunk}\; m) \Downarrow_{k+2} m'}$$

$$\frac{m_v^0 \Downarrow_k m'}{\mathsf{let}\; v.\, \mathsf{in}\; m \Downarrow_{k+1} m'} \qquad \frac{m \Downarrow_{k_1} \lambda.\, n \qquad n_v^0 \Downarrow_{k_2} n'}{\mathsf{app}\; m\; v \Downarrow_{k_1+k_2+1} n'} \qquad \frac{m \Downarrow_{k_1} \mathsf{ret}\; v \qquad n_v^0 \Downarrow_{k_2} n'}{\mathsf{seq}\; m\; n \Downarrow_{k_1+k_2+1} n'}$$

$$\frac{m_1 \Downarrow_{k_1} \mathsf{ret}\; v_1 \qquad m_2 \Downarrow_{k_2} \mathsf{ret}\; v_2 \qquad (n_{v_1}^0)_{v_2}^1 \Downarrow_{k_3} n'}{\mathsf{pseq}\; m_2\; m_1\; n \Downarrow_{k_1+k_2+k_3+1} n'}$$

FIGURE 4.4: Big-Step Semantics for CBPV with Time Cost

CBPV

$$\frac{}{\lambda. m \Downarrow_0 \lambda. m} \qquad \frac{}{\mathsf{ret}\ v \Downarrow_0 \mathsf{ret}\ v} \qquad \frac{m \Downarrow_k m'}{\mathsf{force}\,(\mathsf{thunk}\ m) \Downarrow_{k+2} m'}$$

$$\frac{m^0_v \Downarrow_k m'}{\mathsf{let}\ v.\ \mathsf{in}\ m \Downarrow_{k+1} m'} \qquad \frac{m \Downarrow_{k_1} \lambda. n \qquad n^0_v \Downarrow_{k_2} n'}{\mathsf{app}\ m\ v \Downarrow_{k_1+k_2+1} n'} \qquad \frac{m \Downarrow_{k_1} \mathsf{ret}\ v \qquad n^0_v \Downarrow_{k_2} n'}{\mathsf{seq}\ m\ n \Downarrow_{k_1+k_2+1} n'}$$

$$\frac{m_1 \Downarrow_{k_1} \mathsf{ret}\ v_1 \qquad m_2 \Downarrow_{k_2} \mathsf{ret}\ v_2 \qquad (n^0_{v_1})^1_{v_2} \Downarrow_{k_3} n'}{\mathsf{pseq}\ m_2\ m_1\ n \Downarrow_{k_1+k_2+k_3+1} n'}$$

FIGURE 4.4: Big-Step Semantics for CBPV with Time Cost

CBPV

$$\frac{}{\lambda.\, m \Downarrow_0 \lambda.\, m} \qquad \frac{}{\mathsf{ret}\; v \Downarrow_0 \mathsf{ret}\; v} \qquad \frac{m \Downarrow_k m'}{\mathsf{force}\,(\mathsf{thunk}\; m) \Downarrow_{k+2} m'}$$

$$\frac{m_v^0 \Downarrow_k m'}{\mathsf{let}\; v.\, \mathsf{in}\; m \Downarrow_{k+1} m'} \qquad \frac{m \Downarrow_{k_1} \lambda.\, n \qquad n_v^0 \Downarrow_{k_2} n'}{\mathsf{app}\; m\; v \Downarrow_{k_1+k_2+1} n'} \qquad \frac{m \Downarrow_{k_1} \mathsf{ret}\; v \qquad n_v^0 \Downarrow_{k_2} n'}{\mathsf{seq}\; m\; n \Downarrow_{k_1+k_2+1} n'}$$

$$\frac{m_1 \Downarrow_{k_1} \mathsf{ret}\; v_1 \qquad m_2 \Downarrow_{k_2} \mathsf{ret}\; v_2 \qquad (n_{v_1}^0)_{v_2}^1 \Downarrow_{k_3} n'}{\mathsf{pseq}\; m_2\; m_1\; n \Downarrow_{k_1+k_2+k_3+1} n'}$$

FIGURE 4.4: Big-Step Semantics for CBPV with Time Cost

CBPV

$$\frac{}{\lambda.\, m \Downarrow^{\|\lambda.\, m\|} \lambda.\, m} \qquad \frac{}{\mathsf{ret}\, v \Downarrow^{\|\mathsf{ret}\, v\|} \mathsf{ret}\, v}$$

$$\frac{m \Downarrow^{s} m'}{\mathsf{force}\,(\mathsf{thunk}\, m) \Downarrow^{\max(s,\, \|m\|+2)} m'} \qquad \frac{m^0_v \Downarrow^{s} m'}{\mathsf{let}\, v.\, \mathsf{in}\, m \Downarrow^{\max(s,\, \|v\|+\|m\|+1)} m'}$$

$$\frac{m \Downarrow^{s_1} \lambda.\, n \qquad n^0_v \Downarrow^{s_2} n'}{\mathsf{app}\, m\, v \Downarrow^{\max(s_1+\|n\|+1,\, s_2)} n'} \qquad \frac{m \Downarrow^{s_1} \mathsf{ret}\, v \qquad n^0_v \Downarrow^{s_2} n'}{\mathsf{seq}\, m\, n \Downarrow^{\max(s_1+\|n\|+1,\, s_2)} n'}$$

$$\frac{m_1 \Downarrow^{s_1} \mathsf{ret}\, v_1 \qquad m_2 \Downarrow^{s_2} \mathsf{ret}\, v_2 \qquad (n^0_{v_1})^1_{v2} \Downarrow^{s_3} n'}{\mathsf{pseq}\, m_2\, m_1\, n \Downarrow^{\max(s_1+\|m_2\|+\|n\|+1,\, \max(\|v_1\|+s_2+\|n\|+1, s_3))} m}$$

FIGURE 4.5: Big-Step Semantics for CBPV with Space Cost

CBPV

$$\frac{}{\lambda.\, m \Downarrow^{\|\lambda.\, m\|} \lambda.\, m} \qquad \frac{}{\mathsf{ret}\ v \Downarrow^{\|\mathsf{ret}\ v\|} \mathsf{ret}\ v}$$

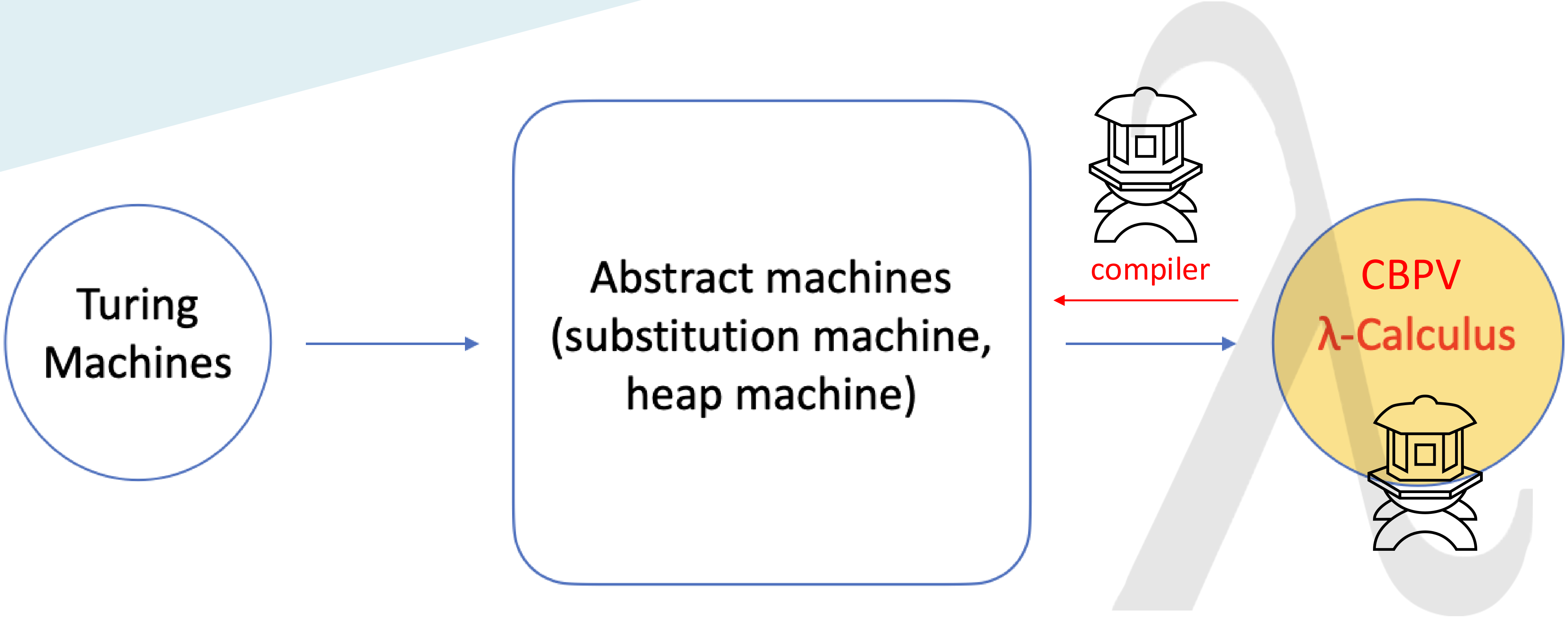
$$\frac{m \Downarrow^{s} m'}{\mathsf{force}\ (\mathsf{thunk}\ m) \Downarrow^{\max(s, \|m\|+2)} m'} \qquad \frac{m_v^0 \Downarrow^{s} m'}{\mathsf{let}\ v.\ \mathsf{in}\ m \Downarrow^{\max(s, \|v\|+\|m\|+1)} m'}$$

$$\frac{m \Downarrow^{s_1} \lambda.\, n \qquad n_v^0 \Downarrow^{s_2} n'}{\mathsf{app}\ m\ v \Downarrow^{\max(s_1+\|n\|+1, s_2)} n'} \qquad \frac{m \Downarrow^{s_1} \mathsf{ret}\ v \qquad n_v^0 \Downarrow^{s_2} n'}{\mathsf{seq}\ m\ n \Downarrow^{\max(s_1+\|n\|+1, s_2)} n'}$$

$$\frac{m_1 \Downarrow^{s_1} \mathsf{ret}\ v_1 \qquad m_2 \Downarrow^{s_2} \mathsf{ret}\ v_2 \qquad (n_{v_1}^0)_{v2}^1 \Downarrow^{s_3} n'}{\mathsf{pseq}\ m_2\ m_1\ n \Downarrow^{\max(s_1+\|m_2\|+\|n\|+1, \max(\|v_1\|+s_2+\|n\|+1, s_3))} m}$$

FIGURE 4.5: Big-Step Semantics for CBPV with Space Cost

A Verified Cost Model for CBPV



Compiler

Definition 36 (Program Tokens).

$$\begin{aligned} \mathsf{t} \in \mathsf{Tok} \quad := \quad & \mathsf{varT}\, x \mid \mathsf{thunkT} \mid \mathsf{endThunkT} \mid \mathsf{forceT} \mid \mathsf{applyT} \mid \\ & \mathsf{lamT} \mid \mathsf{endLamT} \mid \mathsf{retT} \mid \mathsf{endRetT} \mid \mathsf{seqT} \mid \mathsf{endSeqT} \mid \\ & \mathsf{pseqT} \mid \mathsf{endPseqT} \mid \mathsf{letinT} \mid \mathsf{endLetinT} \end{aligned}$$

Compiler

Definition 39 (Compilation Function for Substitution Machine).

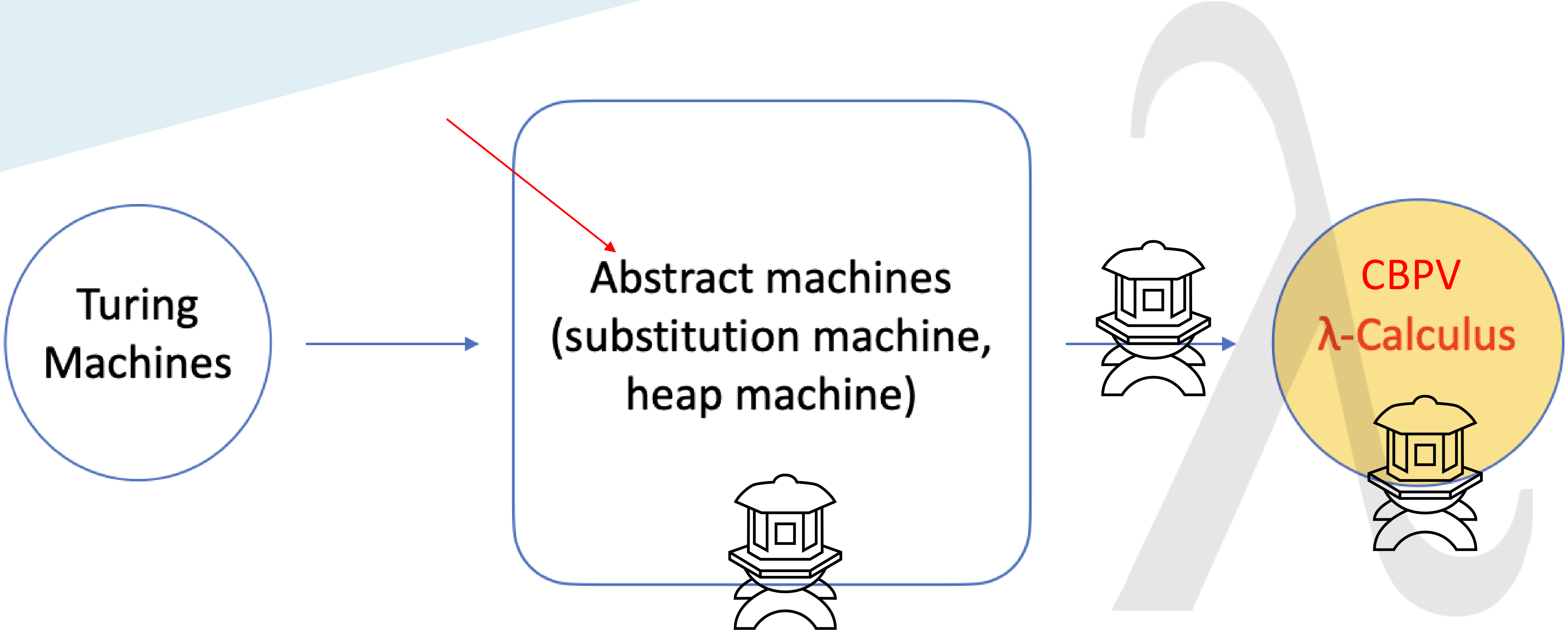
$$
\begin{array}{lcl}
\gamma(\mathsf{var}\ x) & = & \mathsf{varT}\ x \\
\gamma(\mathsf{thunk}\ m) & = & \mathsf{thunkT} :: \gamma(m) +\!\!+ [\mathsf{endThunkT}] \\
\gamma(\mathsf{force}\ v) & = & \gamma(v) +\!\!+ [\mathsf{forceT}] \\
\gamma(\mathsf{ret}\ v) & = & \mathsf{retT} :: \gamma(v) +\!\!+ [\mathsf{endRetT}] \\
\gamma(\lambda.\ m) & = & \mathsf{lamT} :: \gamma(m) +\!\!+ [\mathsf{endLamT}] \\
\gamma(\mathsf{app}\ m\ v) & = & \gamma(m) +\!\!+ \gamma(v) +\!\!+ [\mathsf{applyT}] \\
\gamma(\mathsf{seq}\ m\ n) & = & \gamma(m) +\!\!+ [\mathsf{seqT}] +\!\!+ \gamma(n) +\!\!+ [\mathsf{endSeqT}] \\
\gamma(\mathsf{pseq}\ m_1\ m_2\ n) & = & \gamma(m_1) +\!\!+ \gamma(m_2) +\!\!+ [\mathsf{pseqT}] +\!\!+ \gamma(n) +\!\!+ [\mathsf{endPseqT}] \\
\gamma(\mathsf{let}\ v.\ \mathsf{in}\ m) & = & \gamma(v) +\!\!+ [\mathsf{letinT}] +\!\!+ \gamma(m) +\!\!+ [\mathsf{endLetinT}]
\end{array}
$$

Compiler

Definition 40 (Compilation Function for Heap Machine).

$$
\begin{array}{lcl}
\gamma(\mathsf{var}\ x) & = & \mathsf{varT}\ x \\
\gamma(\mathsf{thunk}\ m) & = & \mathsf{thunkT}\ ::\ \gamma(m) +\!\!+ [\mathsf{endThunkT}] \\
\gamma(\mathsf{force}\ v) & = & \gamma(v) +\!\!+ [\mathsf{forceT}] \\
\gamma(\mathsf{ret}\ v) & = & \gamma(v) +\!\!+ [\mathsf{retT}] \\
\gamma(\lambda.\ m) & = & \mathsf{lamT}\ ::\ \gamma(m) +\!\!+ [\mathsf{endLamT}] \\
\gamma(\mathsf{app}\ m\ v) & = & \gamma(m) +\!\!+ \gamma(v) +\!\!+ [\mathsf{applyT}] \\
\gamma(\mathsf{seq}\ m\ n) & = & \gamma(m) +\!\!+ [\mathsf{seqT}] +\!\!+ \gamma(n) +\!\!+ [\mathsf{endSeqT}] \\
\gamma(\mathsf{pseq}\ m_1\ m_2\ n) & = & \gamma(m_1) +\!\!+ \gamma(m_2) +\!\!+ [\mathsf{pseqT}] +\!\!+ \gamma(n) +\!\!+ [\mathsf{endPseqT}] \\
\gamma(\mathsf{let}\ v.\ \mathsf{in}\ m) & = & \gamma(v) +\!\!+ [\mathsf{letinT}] +\!\!+ \gamma(m) +\!\!+ [\mathsf{endLetinT}]
\end{array}
$$

Abstract Machines



Substitution Machine

Task Stack | Value Stack

Substitution Machine

Definition 33 (Syntax for CBPV).

$$
\begin{array}{llllll}
\textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & & \\
\textit{Computations} & \mathtt{M} & := & \lambda.\,\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid & \mathsf{ret}\ \mathtt{V} \mid \\
& & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M} &
\end{array}
$$

Substitution Machine

Definition 33 (Syntax for CBPV).

$$
\begin{array}{llllll}
\textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & \\
\textit{Computations} & \mathtt{M} & := & \lambda.\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid \quad \mathsf{ret}\ \mathtt{V} \mid \\
 & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M}
\end{array}
$$

Example Transition Rule for Substitution Machine - Variable

$$
\begin{array}{ll|r|}
 & (\mathsf{varT}\ n :: P) :: T & V \\
\triangleright & P ::_{tc} T & \mathsf{varT}\ n :: V
\end{array}
$$

Substitution Machine

Definition 33 (Syntax for CBPV).

$$
\begin{array}{llllll}
\textit{Values} & \mathtt{V} ::= & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & & \\
\textit{Computations} & \mathtt{M} ::= & \lambda.\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid & \mathsf{ret}\ \mathtt{V} \mid \\
 & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M} &
\end{array}
$$

Example Transition Rule for Substitution Machine - Variable

$$
\begin{array}{ll|r|}
 & (\mathsf{varT}\ n :: P) :: T & V \\
\triangleright & P ::_{tc} T & \mathsf{varT}\ n :: V
\end{array}
$$

Lemma 45 (Substitution Machine Time Simulation). *If $m \Downarrow_k n$, then there exists k' such that $(P_m, []) \triangleright_{k'} ([], P_n)$ where $k' \leq 3 * k + 1$ and $P_m \gg m$ and $P_n \gg n$.*

Lemma 46 (Substitution Machine Space Simulation). *If $m \Downarrow^s n$ then there exists σ such that $(P_m, []) \triangleright_*^\sigma ([], P_n)$, where $s \leq \sigma \leq 9 * s$ and $P_m \gg m$ and $P_n \gg n$.*

Substitution Machine

Definition 33 (Syntax for CBPV).

$$\begin{array}{llllll} \textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & \\ \textit{Computations} & \mathtt{M} & := & \lambda.\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid \quad \mathsf{ret}\ \mathtt{V} \mid \\ & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M} \end{array}$$

Example Transition Rule for Substitution Machine - Variable

$$\begin{array}{ll|r|} & (\mathsf{varT}\ n :: P) :: T & V \\ \triangleright & P ::_{tc} T & \mathsf{varT}\ n :: V \end{array}$$

Lemma 45 (Substitution Machine Time Simulation). *If $m \Downarrow_k n$, then there exists k' such that $(P_m, []) \triangleright_{k'} ([], P_n)$ where $k' \leq 3 * k + 1$ and $P_m \gg m$ and $P_n \gg n$.*

Lemma 46 (Substitution Machine Space Simulation). *If $m \Downarrow^s n$ then there exists σ such that $(P_m, []) \triangleright^{\sigma}_{*} ([], P_n)$, where $s \leq \sigma \leq 9 * s$ and $P_m \gg m$ and $P_n \gg n$.*

Heap Machine

Task Stack | Value Stack | Heap

Heap Machine

Definition 33 (Syntax for CBPV).

$$
\begin{array}{llllll}
\textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & & \\
\textit{Computations} & \mathtt{M} & := & \lambda.\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid & \mathsf{ret}\ \mathtt{V} \mid \\
 & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M} &
\end{array}
$$

Example Transition Rule for Substitution Machine - Application

$$
\begin{array}{rl|r|c|l}
 & \langle \mathsf{applyT} :: P, a\rangle :: T & Q :: \langle \mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}], b\rangle :: V & H & \mathsf{put}\ H\{Q, b\} = (H', c) \\
\triangleright & \langle M, c\rangle :: \langle P, a\rangle :: T & V & H' &
\end{array}
$$

Heap Machine

Definition 33 (Syntax for CBPV).

Values	$\mathtt{V}$	$:=$	$\mathsf{var}\ x \mid$	$\mathsf{thunk}\ \mathtt{M}$		
Computations	$\mathtt{M}$	$:=$	$\lambda.\mathtt{M} \mid$	$\mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid$	$\mathsf{force}\ \mathtt{V} \mid$	$\mathsf{ret}\ \mathtt{V} \mid$
			$\mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid$	$\mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid$	$\mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M}$	

Example Transition Rule for Substitution Machine - Variable

$\langle \mathsf{varT}\ x :: P,\ a \rangle :: T$	V	H	$\mathsf{lookup}\ H\ a\ x = g$
$\triangleright\ \langle P,\ a \rangle :: T$	$g :: V$	H	

Heap Machine

Definition 33 (Syntax for CBPV).

$$\begin{array}{llllll} \textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & \\ \textit{Computations} & \mathtt{M} & := & \lambda.\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid \quad \mathsf{ret}\ \mathtt{V} \mid \\ & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M} \end{array}$$

Example Transition Rule for Substitution Machine - Variable

$\langle \mathsf{varT}\ x :: P, a\rangle :: T$	V	H	$\mathsf{lookup}\ H\ a\ x = g$
$\rhd\ \langle P, a\rangle :: T$	$g :: V$	H	

Lemma 55 (Heap Machine Time Simulation). *If $m \Downarrow_k n$ then there exists k' such that $(\langle P_m, 0\rangle, [\,], [\,]) \rhd_{k'} ([\,], \langle P_n, a\rangle, H)$, where $k' \leq 7 * k + 5$ and $\langle P_m, 0\rangle \gg m$ and $\langle P_n, a\rangle \gg_H n$.*

Lemma 56 (Heap Machine Space Simulation). *If $(P_m, [\,], [\,]) \rhd_k (T, N, H)$ and $P_m \gg m$ then $\|T \mathbin{+\!\!+} N \mathbin{+\!\!+} H\| \leq (k+1) * (3 * k + 4 * \|m\|)$.*

Heap Machine

Definition 33 (Syntax for CBPV).

$$
\begin{array}{llllll}
\textit{Values} & \mathtt{V} & := & \mathsf{var}\ x \mid & \mathsf{thunk}\ \mathtt{M} & \\
\textit{Computations} & \mathtt{M} & := & \lambda.\mathtt{M} \mid & \mathsf{app}\ \mathtt{M}\ \mathtt{V} \mid & \mathsf{force}\ \mathtt{V} \mid \quad \mathsf{ret}\ \mathtt{V} \mid \\
 & & & \mathsf{seq}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{pseq}\ \mathtt{M}\ \mathtt{M}\ \mathtt{M} \mid & \mathsf{let}\ \mathtt{V}.\ \mathsf{in}\ \mathtt{M}
\end{array}
$$

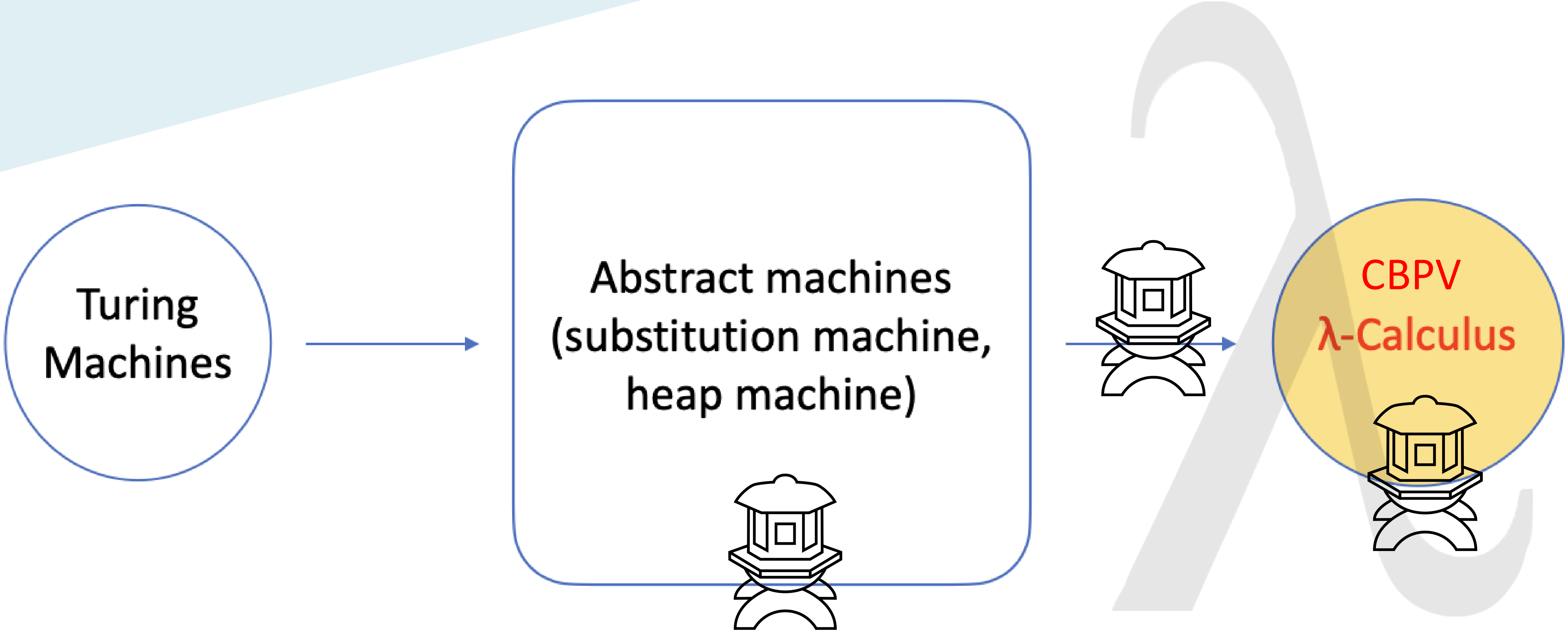
Example Transition Rule for Substitution Machine - Variable

$\langle \mathsf{varT}\ x :: P, a\rangle :: T$ $\triangleright\ \langle P, a\rangle :: T$	V $g :: V$	H H	$\mathsf{lookup}\ H\ a\ x = g$

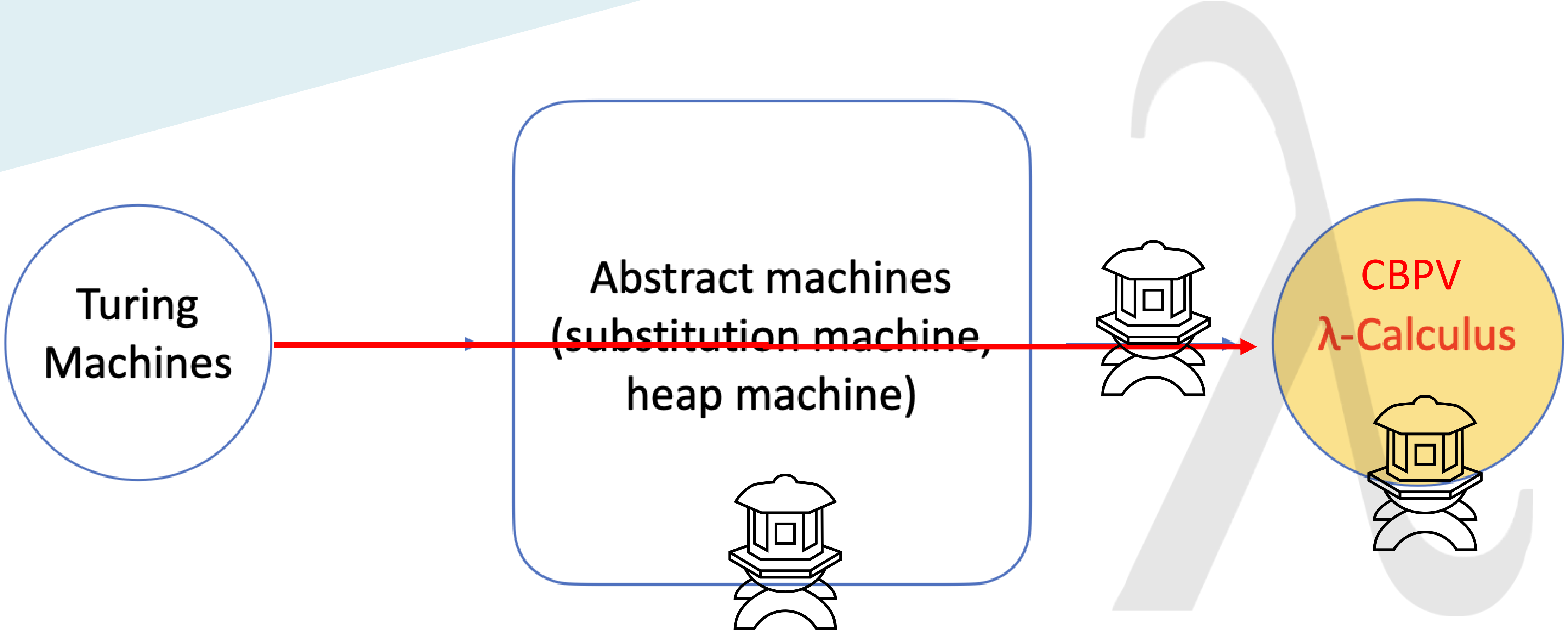
Lemma 55 (Heap Machine Time Simulation). *If $m \Downarrow_k n$ then there exists k' such that $(\langle P_m, 0\rangle, [\,], [\,]) \triangleright_{k'} ([\,], \langle P_n, a\rangle, H)$, where $k' \leq 7 * k + 5$ and $\langle P_m, 0\rangle \gg m$ and $\langle P_n, a\rangle \gg_H n$.*

Lemma 56 (Heap Machine Space Simulation). *If $(P_m, [\,], [\,]) \triangleright_k (T, N, H)$ and $P_m \gg m$ then $\|T +\!\!+ N +\!\!+ H\| \leq (k+1) * (3 * k + 4 * \|m\|)$.*

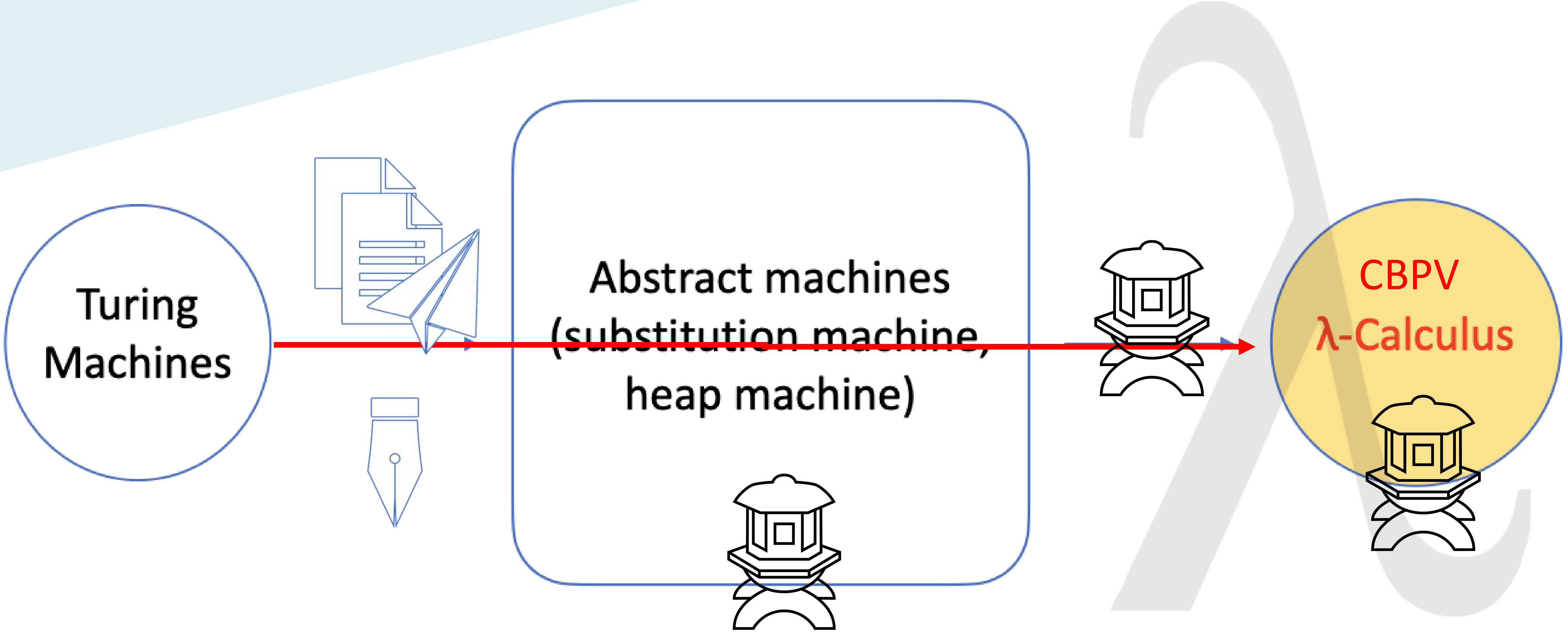
A Verified Cost Model for CBPV



A Verified Cost Model for CBPV



A Verified Cost Model for CBPV



Turing Machine Simulating CBPV

Task Stack	Value Stack	Assumption
$(\mathsf{varT}\ n :: P) :: T$	V	
$\triangleright\ P ::_{tc} T$	$\mathsf{varT}\ n :: V$	
$(\mathsf{thunkT} :: P) :: T$	V	$\varphi P = (M, Q)$
$\triangleright\ Q ::_{tc} T$	$\mathsf{thunkT} :: M \mathbin{+\!\!+} [\mathsf{endThunkT}] :: V$	
$(\mathsf{forceT} :: P) :: T$	$(\mathsf{thunkT} :: K) :: V$	$\varphi K = (M, [\,])$
$\triangleright\ (M \mathbin{+\!\!+} P) ::_{tc} T$	V	
$(\mathsf{lamT} :: P) :: T$	V	$\varphi P = (M, Q)$
$\triangleright\ Q ::_{tc} T$	$(\mathsf{lamT} :: M \mathbin{+\!\!+} [\mathsf{endLamT}]) :: V$	
$(\mathsf{applyT} :: P) :: T$	$Q :: (\mathsf{lamT} :: M \mathbin{+\!\!+} [\mathsf{endLamT}]) :: V$	
$\triangleright\ M^0_Q :: (P ::_{tc} T)$	V	
$(\mathsf{retT} :: P) :: T$	V	$\varphi P = (U, Q)$
$\triangleright\ Q ::_{tc} T$	$(\mathsf{retT} :: U \mathbin{+\!\!+} [\mathsf{endRetT}]) :: V$	
$(\mathsf{seqT} :: P) :: T$	$(\mathsf{retT} :: U \mathbin{+\!\!+} [\mathsf{endRetT}]) :: V$	$\varphi P = (N, Q)$
$\triangleright\ N^0_U :: (Q ::_{tc} T)$	V	
$(\mathsf{pseqT} :: P) :: T$	$(\mathsf{retT} :: U_2 \mathbin{+\!\!+} [\mathsf{endRetT}]) :: (\mathsf{retT} :: U_1 \mathbin{+\!\!+} [\mathsf{endRetT}]) :: V$	$\varphi P = (N, Q)$
$\triangleright\ (N^0_{U_1})^1_{U_2} :: (Q ::_{tc} T)$	V	
$(\mathsf{letinT} :: P) :: T$	$K :: V$	$\varphi P = (M, Q)$
$\triangleright\ M^0_K :: (Q ::_{tc} T)$	V	

FIGURE 4.6: Transition Rules for the Substitution Machine

Turing Machine Simulating CBPV

Task Stack	Value Stack	Assumption
$(\mathsf{varT}\ n :: P) :: T$ $\triangleright\ P ::_{tc} T$	V $\mathsf{varT}\ n :: V$	
$(\mathsf{thunkT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $\mathsf{thunkT} :: M +\!\!+ [\mathsf{endThunkT}] :: V$	$\varphi P = (M, Q)$
$(\mathsf{forceT} :: P) :: T$ $\triangleright\ (M +\!\!+ P) ::_{tc} T$	$(\mathsf{thunkT} :: K) :: V$ V	$\varphi K = (M, [\,])$
$(\mathsf{lamT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $(\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$	$\varphi P = (M, Q)$
$(\mathsf{applyT} :: P) :: T$ $\triangleright\ M^0_Q :: (P ::_{tc} T)$	$Q :: (\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$ V	
$(\mathsf{retT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$	$\varphi P = (U, Q)$
$(\mathsf{seqT} :: P) :: T$ $\triangleright\ N^0_U :: (Q ::_{tc} T)$	$(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$ V	$\varphi P = (N, Q)$
$(\mathsf{pseqT} :: P) :: T$ $\triangleright\ (N^0_{U_1})^1_{U_2} :: (Q ::_{tc} T)$	$(\mathsf{retT} :: U_2 +\!\!+ [\mathsf{endRetT}]) :: (\mathsf{retT} :: U_1 +\!\!+ [\mathsf{endRetT}]) :: V$ V	$\varphi P = (N, Q)$
$(\mathsf{letinT} :: P) :: T$ $\triangleright\ M^0_K :: (Q ::_{tc} T)$	$K :: V$ V	$\varphi P = (M, Q)$

M_1

FIGURE 4.6: Transition Rules for the Substitution Machine

Turing Machine Simulating CBPV

Task Stack	Value Stack	Assumption
$(\mathsf{varT}\ n :: P) :: T$ $\triangleright\ P ::_{tc} T$	V $\mathsf{varT}\ n :: V$	
$(\mathsf{thunkT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $\mathsf{thunkT} :: M +\!\!+ [\mathsf{endThunkT}] :: V$	$\varphi P = (M, Q)$
$(\mathsf{forceT} :: P) :: T$ $\triangleright\ (M +\!\!+ P) ::_{tc} T$	$(\mathsf{thunkT} :: K) :: V$ V	$\varphi K = (M, [\,])$
$(\mathsf{lamT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $(\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$	$\varphi P = (M, Q)$
$(\mathsf{applyT} :: P) :: T$ $\triangleright\ M^0_Q :: (P ::_{tc} T)$	$Q :: (\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$ V	
$(\mathsf{retT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$	$\varphi P = (U, Q)$
$(\mathsf{seqT} :: P) :: T$ $\triangleright\ N^0_U :: (Q ::_{tc} T)$	$(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$ V	$\varphi P = (N, Q)$
$(\mathsf{pseqT} :: P) :: T$ $\triangleright\ (N^0_{U_1})^1_{U_2} :: (Q ::_{tc} T)$	$(\mathsf{retT} :: U_2 +\!\!+ [\mathsf{endRetT}]) :: (\mathsf{retT} :: U_1 +\!\!+ [\mathsf{endRetT}]) :: V$ V	$\varphi P = (N, Q)$
$(\mathsf{letinT} :: P) :: T$ $\triangleright\ M^0_K :: (Q ::_{tc} T)$	$K :: V$ V	$\varphi P = (M, Q)$

M_1

M_2

FIGURE 4.6: Transition Rules for the Substitution Machine

Turing Machine Simulating CBPV

Task Stack	Value Stack	Assumption
$(\mathsf{varT}\ n :: P) :: T$	V	
▷ $P ::_{tc} T$	$\mathsf{varT}\ n :: V$	
$(\mathsf{thunkT} :: P) :: T$	V	$\varphi P = (M, Q)$
▷ $Q ::_{tc} T$	$\mathsf{thunkT} :: M +\!\!+ [\mathsf{endThunkT}] :: V$	
$(\mathsf{forceT} :: P) :: T$	$(\mathsf{thunkT} :: K) :: V$	$\varphi K = (M, [\,])$
▷ $(M +\!\!+ P) ::_{tc} T$	V	
$(\mathsf{lamT} :: P) :: T$	V	$\varphi P = (M, Q)$
▷ $Q ::_{tc} T$	$(\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$	
$(\mathsf{applyT} :: P) :: T$	$Q :: (\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$	
▷ $M^0_Q :: (P ::_{tc} T)$	V	
$(\mathsf{retT} :: P) :: T$	V	$\varphi P = (U, Q)$
▷ $Q ::_{tc} T$	$(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$	
$(\mathsf{seqT} :: P) :: T$	$(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$	$\varphi P = (N, Q)$
▷ $N^0_U :: (Q ::_{tc} T)$	V	
$(\mathsf{pseqT} :: P) :: T$	$(\mathsf{retT} :: U_2 +\!\!+ [\mathsf{endRetT}]) :: (\mathsf{retT} :: U_1 +\!\!+ [\mathsf{endRetT}]) :: V$	$\varphi P = (N, Q)$
▷ $(N^0_{U_1})^1_{U_2} :: (Q ::_{tc} T)$	V	
$(\mathsf{letinT} :: P) :: T$	$K :: V$	$\varphi P = (M, Q)$
▷ $M^0_K :: (Q ::_{tc} T)$	V	

FIGURE 4.6: Transition Rules for the Substitution Machine

M_1

M_2

M_...

Turing Machine Simulating CBPV

Task Stack	Value Stack	Assumption
$(\mathsf{varT}\ n :: P) :: T$ $\triangleright\ P ::_{tc} T$	V $\mathsf{varT}\ n :: V$	
$(\mathsf{thunkT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $\mathsf{thunkT} :: M +\!\!+ [\mathsf{endThunkT}] :: V$	$\varphi P = (M, Q)$
$(\mathsf{forceT} :: P) :: T$ $\triangleright\ (M +\!\!+ P) ::_{tc} T$	$(\mathsf{thunkT} :: K) :: V$ V	$\varphi K = (M, [\,])$
$(\mathsf{lamT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $(\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$	$\varphi P = (M, Q)$
$(\mathsf{applyT} :: P) :: T$ $\triangleright\ M^0_Q :: (P ::_{tc} T)$	$Q :: (\mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}]) :: V$ V	
$(\mathsf{retT} :: P) :: T$ $\triangleright\ Q ::_{tc} T$	V $(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$	$\varphi P = (U, Q)$
$(\mathsf{seqT} :: P) :: T$ $\triangleright\ N^0_U :: (Q ::_{tc} T)$	$(\mathsf{retT} :: U +\!\!+ [\mathsf{endRetT}]) :: V$ V	$\varphi P = (N, Q)$
$(\mathsf{pseqT} :: P) :: T$ $\triangleright\ (N^0_{U_1})^1_{U_2} :: (Q ::_{tc} T)$	$(\mathsf{retT} :: U_2 +\!\!+ [\mathsf{endRetT}]) :: (\mathsf{retT} :: U_1 +\!\!+ [\mathsf{endRetT}]) :: V$ V	$\varphi P = (N, Q)$
$(\mathsf{letinT} :: P) :: T$ $\triangleright\ M^0_K :: (Q ::_{tc} T)$	$K :: V$ V	$\varphi P = (M, Q)$

FIGURE 4.6: Transition Rules for the Substitution Machine

M_{subst}

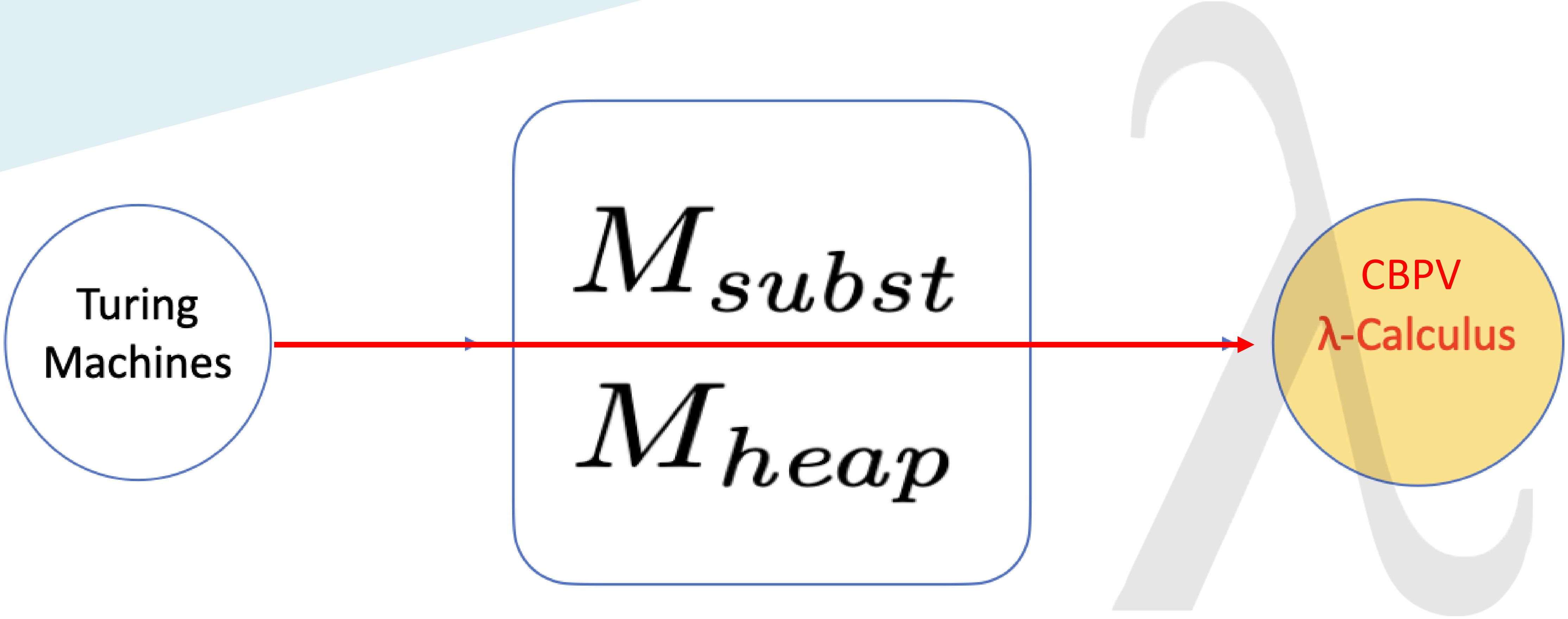
Turing Machine Simulating CBPV

	Task Stack	Value Stack	Heap	Assumption
	$\langle \mathsf{varT}\ x :: P,\ a\rangle :: T$	V	H	lookup $H\ a\ x = g$
$\triangleright$	$\langle P,\ a\rangle :: T$	$g :: V$	H	
	$\langle \mathsf{thunkT} :: P,\ a\rangle :: T$	V	H	$\varphi P = (M, Q)$
$\triangleright$	$\langle Q,\ a\rangle :: T$	$\langle \mathsf{thunkT} :: M +\!\!+ [\mathsf{endThunkT}],\ a\rangle :: V$	H	
	$\langle \mathsf{forceT} :: P,\ a\rangle :: T$	$\langle \mathsf{thunkT} :: M +\!\!+ [\mathsf{endThunkT}],\ b\rangle :: V$	H	
$\triangleright$	$\langle M,\ b\rangle :: \langle P,\ a\rangle :: T$	V	H	
	$\langle \mathsf{lamT} :: P,\ a\rangle :: T$	V	H	$\varphi P = (M, Q)$
$\triangleright$	$\langle Q,\ a\rangle :: T$	$\langle \mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}],\ a\rangle :: V$	H	
	$\langle \mathsf{applyT} :: P,\ a\rangle :: T$	$Q :: \langle \mathsf{lamT} :: M +\!\!+ [\mathsf{endLamT}],\ b\rangle :: V$	H	put $H\{Q,\ b\} = (H', c)$
$\triangleright$	$\langle M,\ c\rangle :: \langle P,\ a\rangle :: T$	V	H'	
	$\langle \mathsf{retT} :: P,\ a\rangle :: T$	$\langle M,\ b\rangle :: V$	H	
$\triangleright$	$\langle P,\ a\rangle :: T$	$\langle M,\ b\rangle :: V$	H	
				$\varphi P = (N, Q)$
	$\langle \mathsf{seqT} :: P,\ a\rangle :: T$	$\langle M,\ b\rangle :: V$	H	put $H\{\langle M,\ b\rangle,\ a\} = (H', c)$
$\triangleright$	$\langle N,\ c\rangle :: \langle Q,\ a\rangle :: T$	V	H'	
				$\varphi P = (N, Q)$
				put $H\{\langle M_2,\ b_2\rangle,\ a\} = (H_1, c_1)$
	$\langle \mathsf{pseqT} :: P,\ a\rangle :: T$	$\langle M_2,\ b_2\rangle :: \langle M_1,\ b_1\rangle :: V$	H	put $H_1\{\langle M_1,\ b_1\rangle,\ a\} = (H_2, c_2)$
$\triangleright$	$\langle N,\ c_2\rangle :: \langle Q,\ a\rangle :: T$	V	H_2	
				$\varphi P = (M, Q)$
	$\langle \mathsf{letinT} :: P,\ a\rangle :: T$	$K :: V$	H	put $H\{K,\ a\} = (H', b)$
$\triangleright$	$\langle M,\ b\rangle :: \langle Q,\ a\rangle :: T$	V	H'	
	$\langle [\,],\ a\rangle :: T$	V	H	
$\triangleright$	T	V	H	

FIGURE 4.8: Transition Rules for the Heap Machine

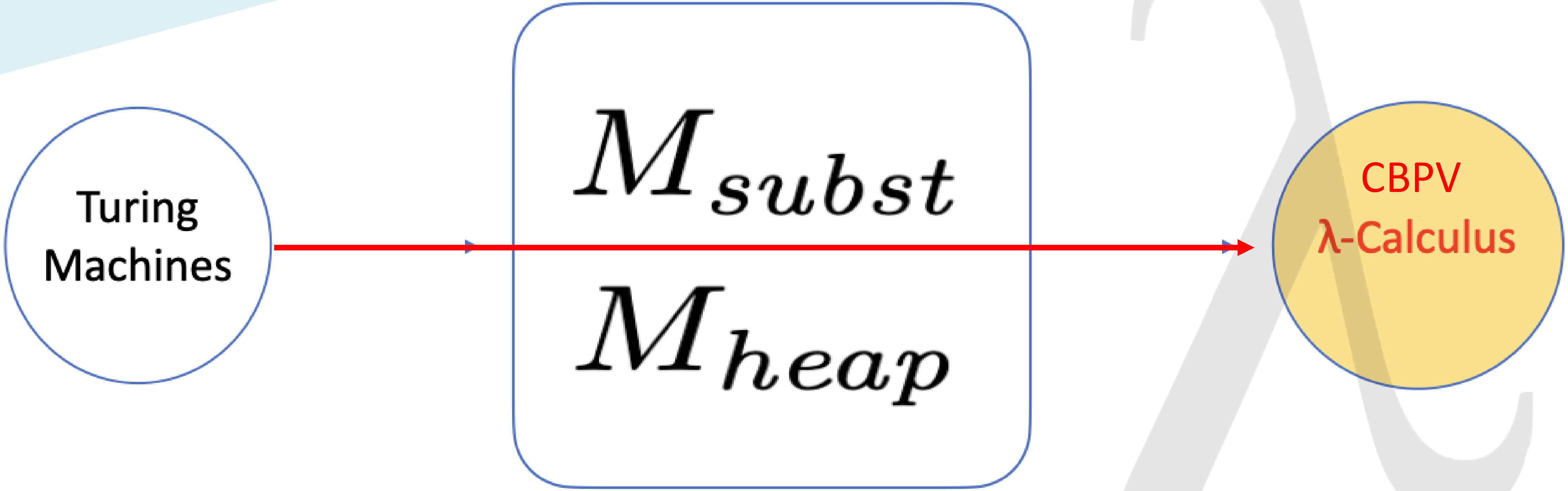
$\rightarrow$ M_{heap}

Turing Machine Simulating CBPV

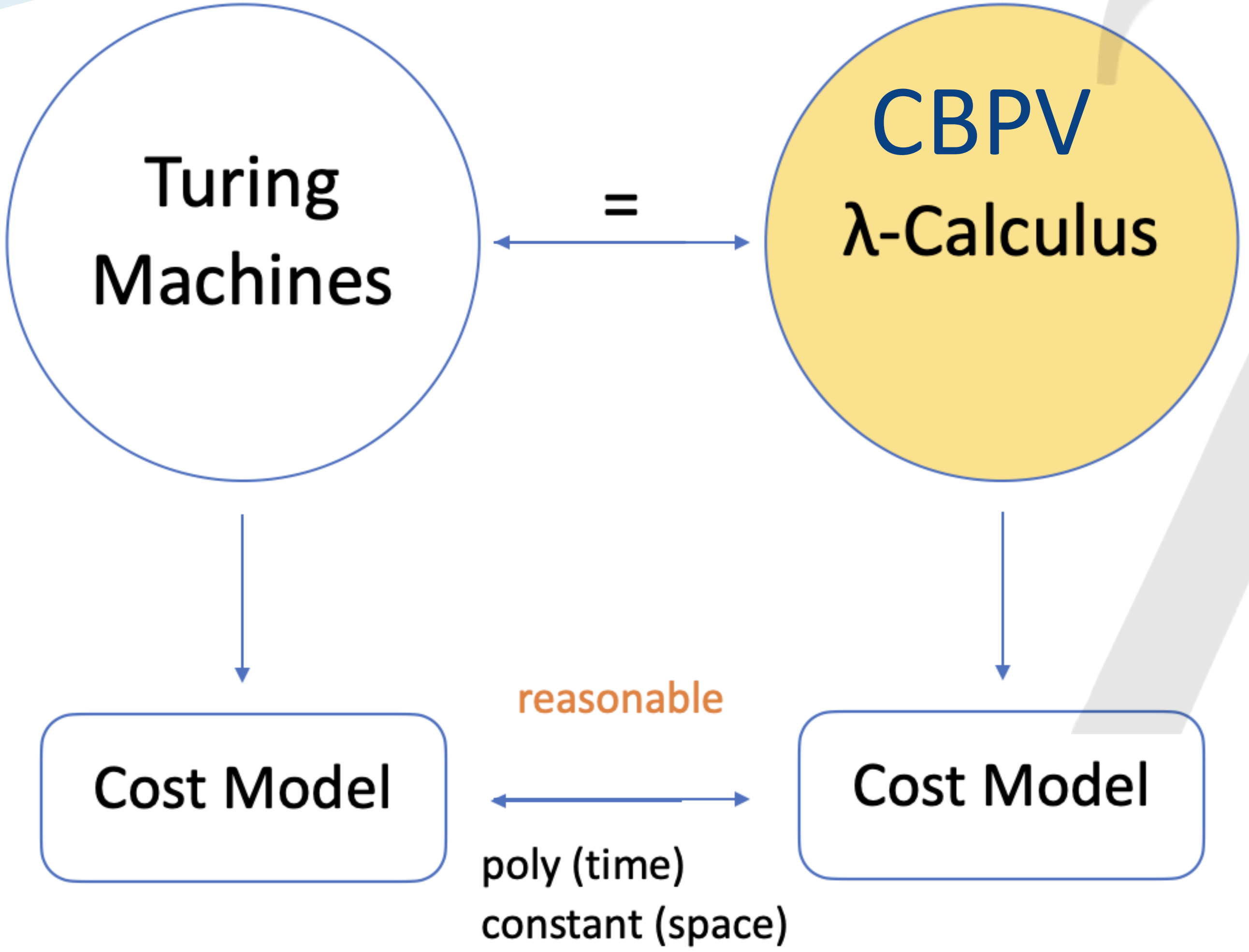


Turing Machine Simulating CBPV

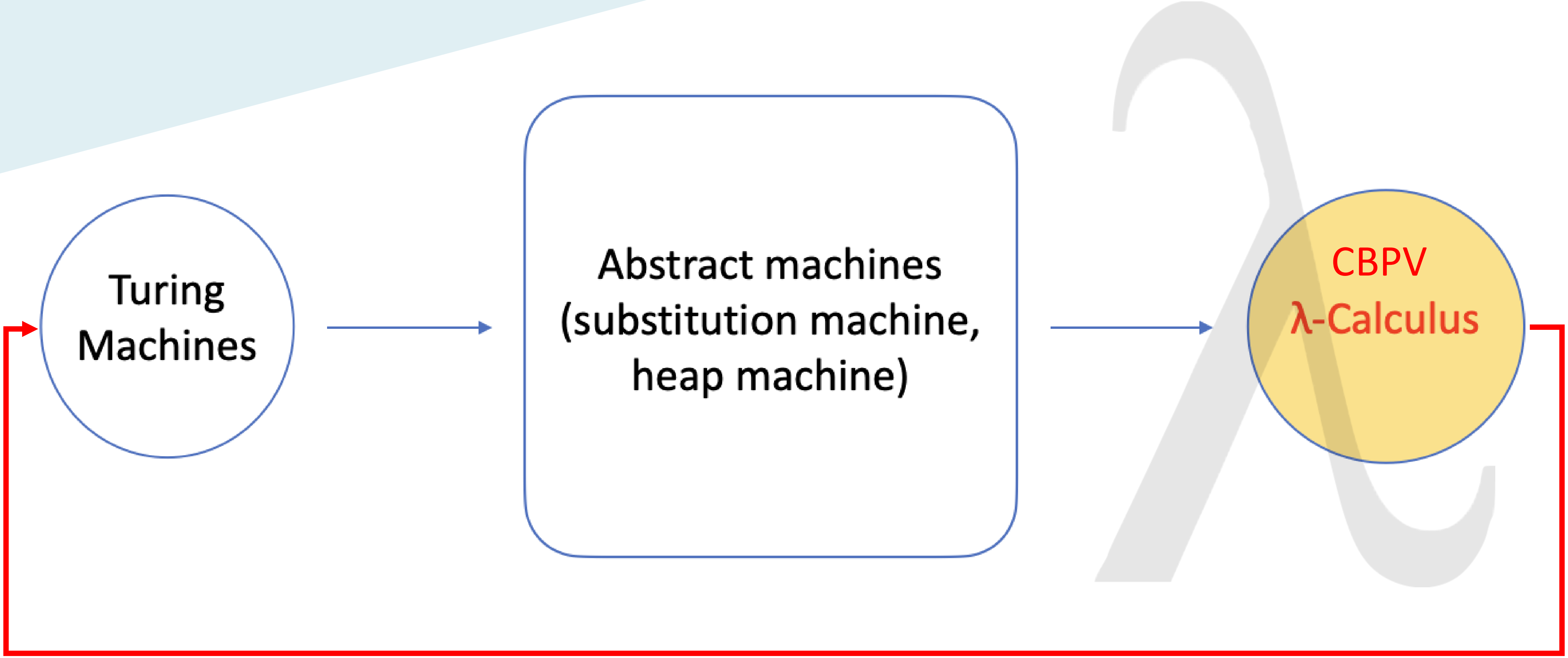
Alternating simulation strategy: [Forster, Kunze, and Roth 2020]



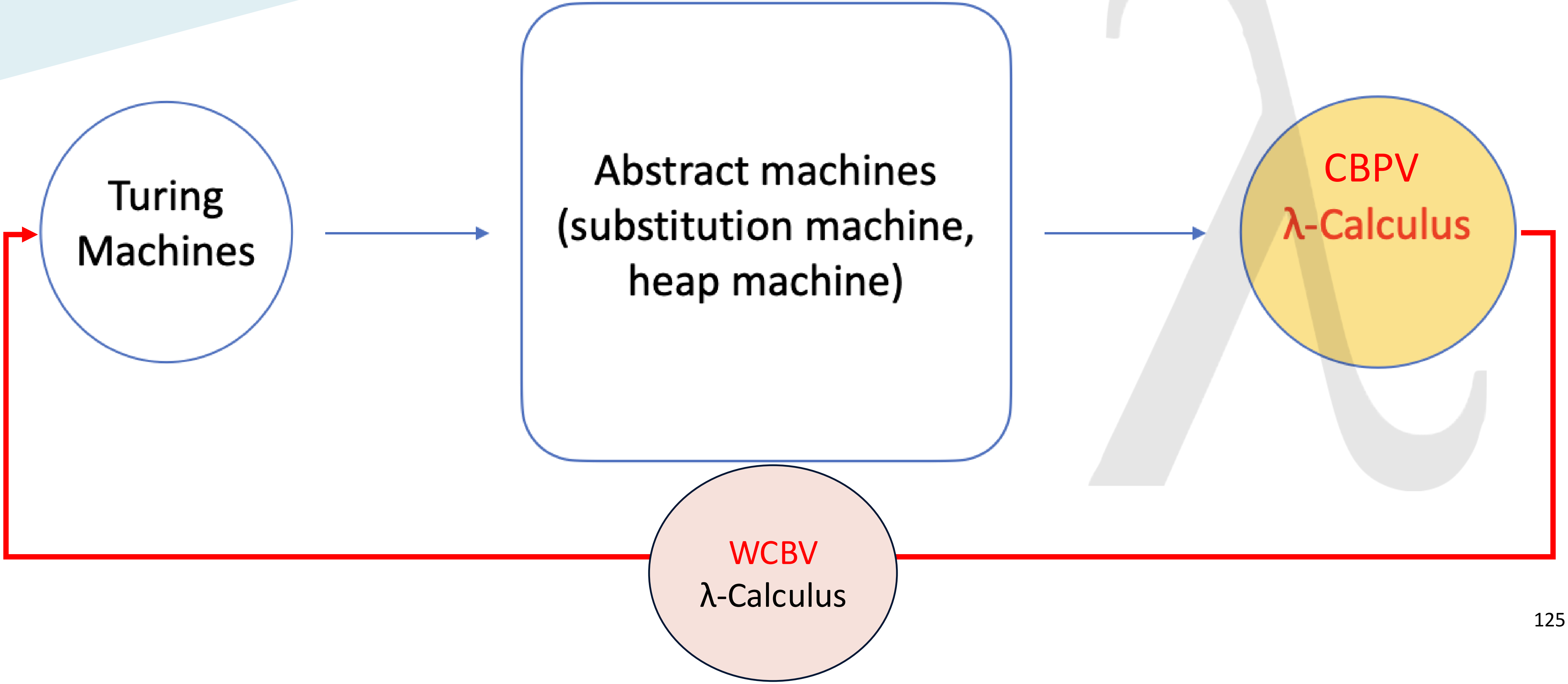
A Verified Cost Model for CBPV



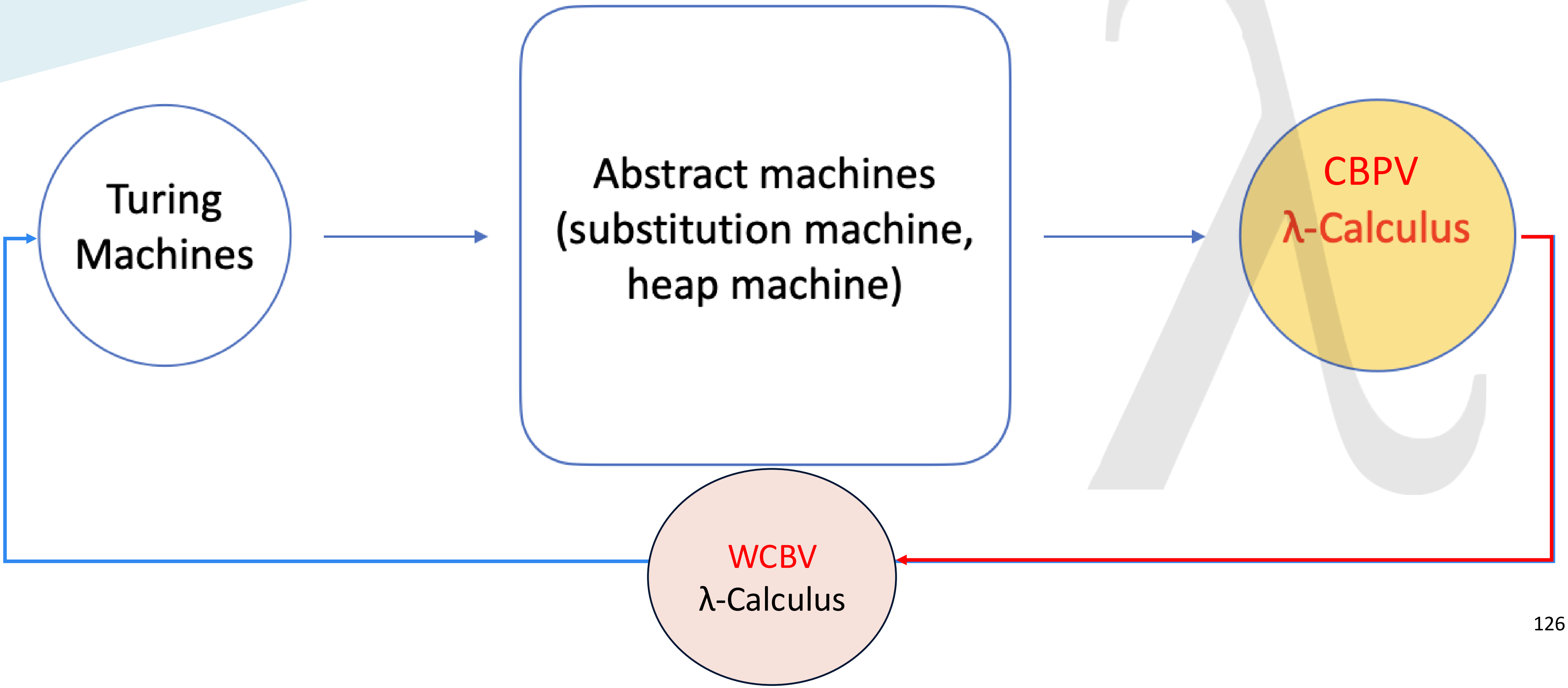
CBPV Simulating Turing Machines



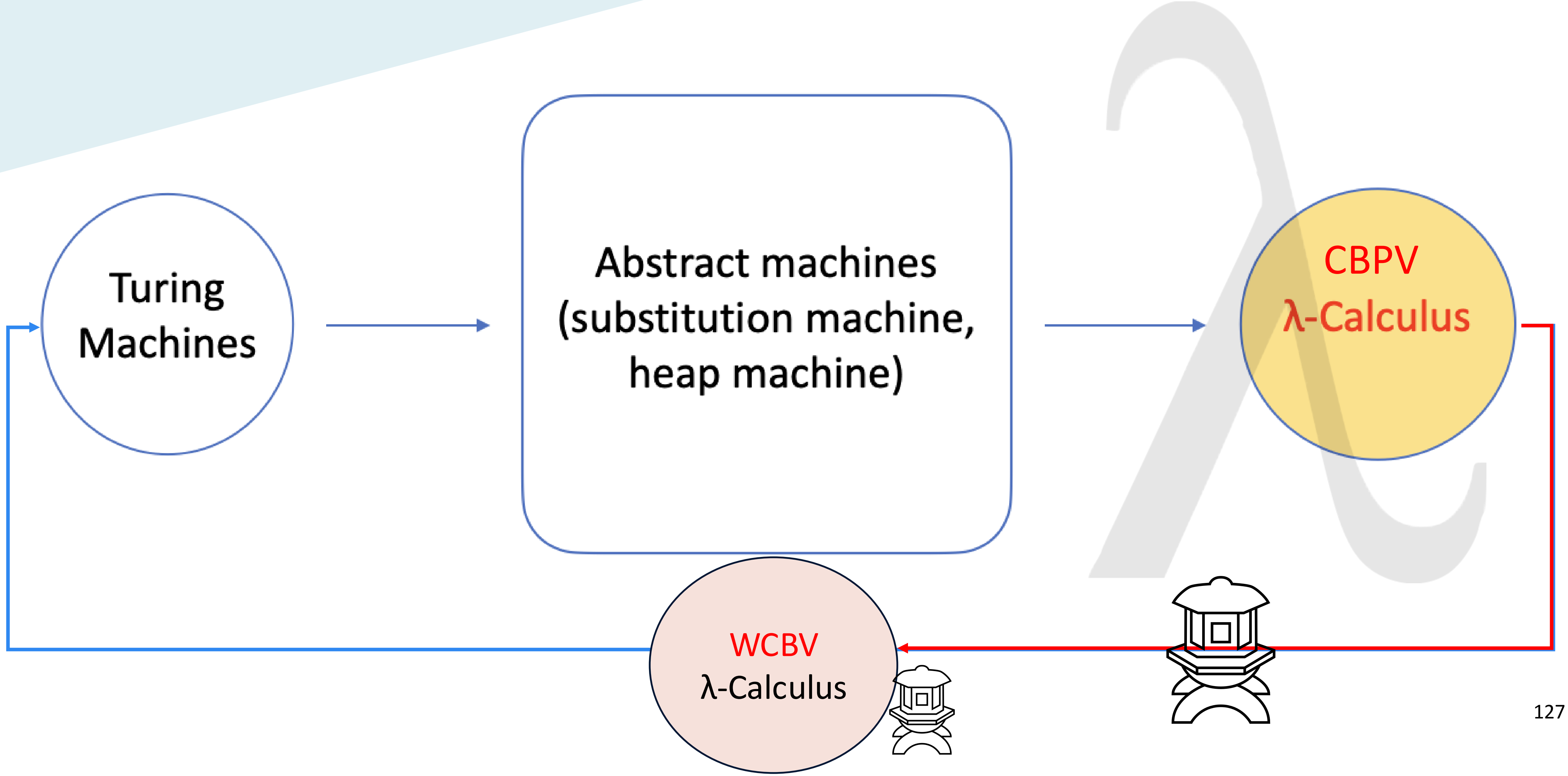
CBPV Simulating Turing Machines



CBPV Simulating Turing Machines



CBPV Simulating Turing Machines



CBPV Simulating WCBV

Definition 18 (Syntax for WCBV). *n is a constant.*

$$s, t, u, v = \mathsf{var}\ n \mid \mathsf{app}\ s\ t \mid \lambda.\ s$$

CBPV Simulating WCBV

Definition 18 (Syntax for WCBV). *n is a constant.*

$$s, t, u, v = \mathsf{var}\, n \,|\, \mathsf{app}\, s\, t \,|\, \lambda.\, s$$

Compilation Function:

$$c(\mathsf{var}\, x) \;=\; \mathsf{var}\, x \qquad c(\lambda.\, t) \;=\; \mathsf{ret\, thunk}\, \lambda.\, c(t)$$

$$c(\mathsf{app}\, t\, u) \;=\; \mathsf{pseq}\, (c(u))\, (c(t))\, (\mathsf{app}\, (\mathsf{force\, var}\, 0)\, (\mathsf{var}\, 1))$$

CBPV Simulating WCBV

Definition 18 (Syntax for WCBV). *n is a constant.*

$$s, t, u, v = \mathsf{var}\ n \mid \mathsf{app}\ s\ t \mid \lambda. s$$

Compilation Function:

$$\begin{array}{lcllcl} c(\mathsf{var}\ x) & = & \mathsf{var}\ x & c(\lambda. t) & = & \mathsf{ret\ thunk}\ \lambda. c(t) \\ c(\mathsf{app}\ t\ u) & = & \multicolumn{4}{l}{\mathsf{pseq}\ (c(u))\ (c(t))\ (\mathsf{app}\ (\mathsf{force\ var}\ 0)\ (\mathsf{var}\ 1))} \end{array}$$

Theorem 62. *For each closed WCBV term t, we have:*

1. *If $t \Downarrow_k u$, then there exists k' such that $c(t) \Downarrow_{k'} c(u)$ and $k' \leq 5 * k$; and*
2. *If $t \Downarrow^s u$, then there exists s' such that $c(t) \Downarrow^{s'} c(u)$ and $s' \leq 6 * s$.*

CBPV Simulating WCBV

Definition 18 (Syntax for WCBV). *n is a constant.*

$$s, t, u, v = \mathsf{var}\, n | \mathsf{app}\, s\, t | \lambda. s$$

Compilation Function:

$$\begin{aligned} c(\mathsf{var}\, x) &= \mathsf{var}\, x & c(\lambda. t) &= \mathsf{ret}\,\mathsf{thunk}\, \lambda. c(t) \\ c(\mathsf{app}\, t\, u) &= \mathsf{pseq}\, (c(u))\, (c(t))\, (\mathsf{app}\, (\mathsf{force}\,\mathsf{var}\, 0)\, (\mathsf{var}\, 1)) \end{aligned}$$

Theorem 62. *For each closed WCBV term t, we have:*

1. *If $t \Downarrow_k u$, then there exists k' such that $c(t) \Downarrow_{k'} c(u)$ and $k' \leq 5 * k$; and*
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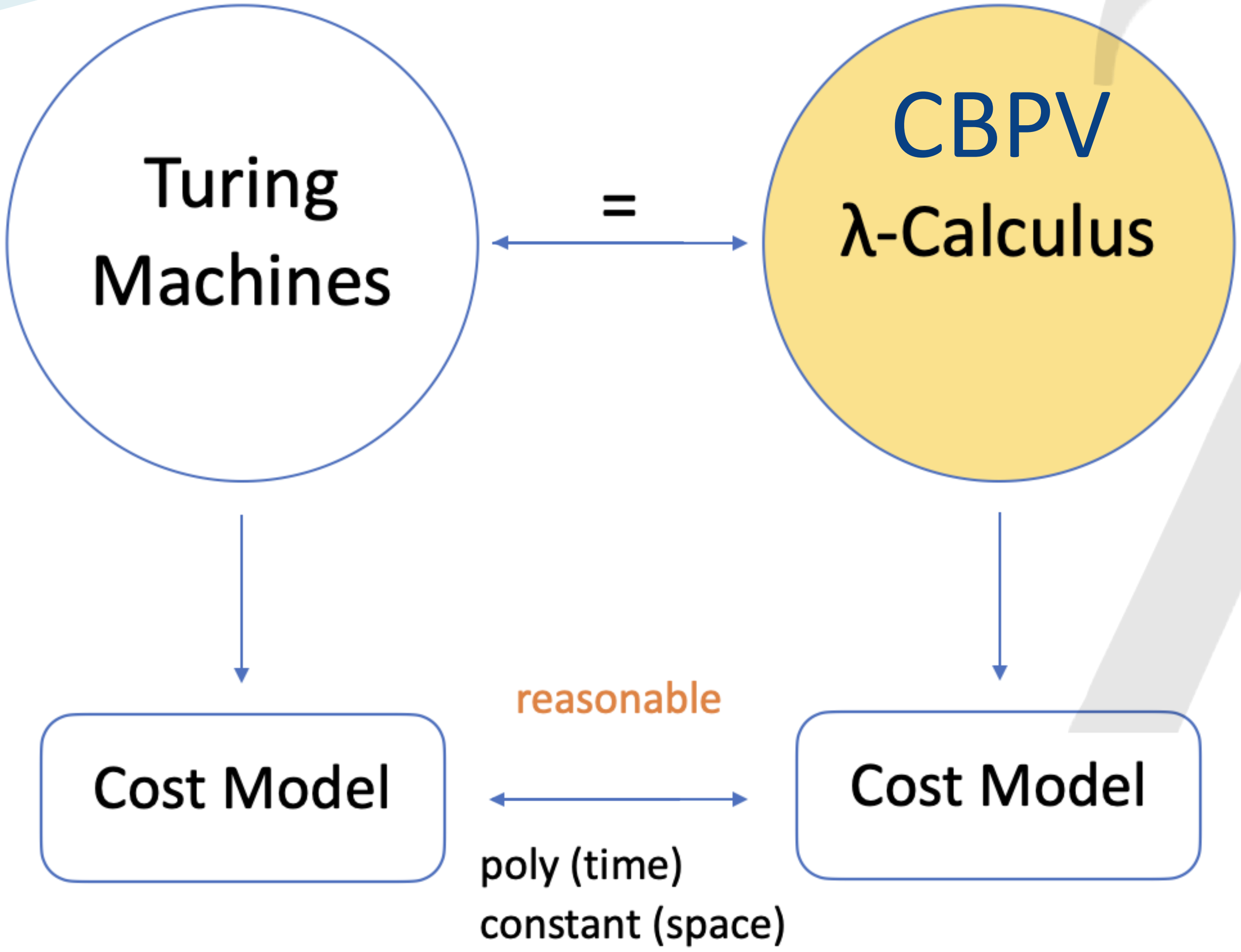
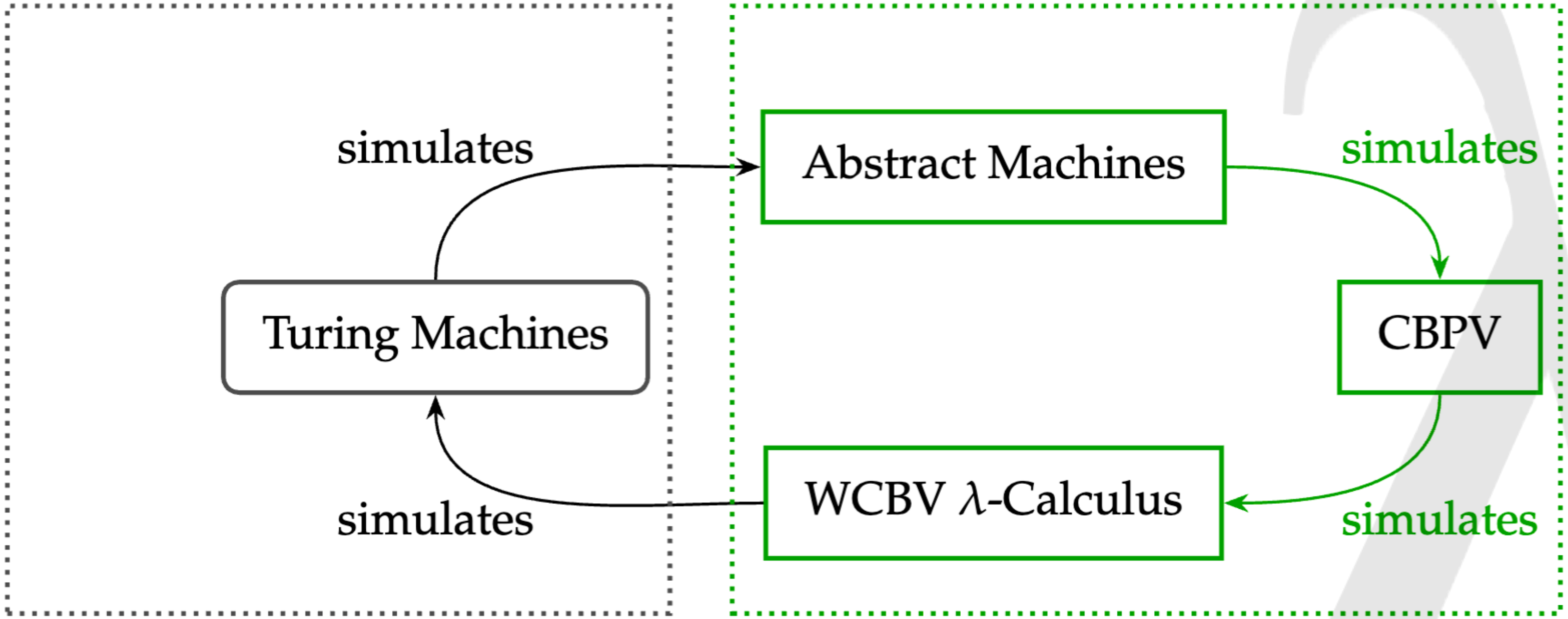
A Verified Cost Model for CBPV



A Verified Cost Model for CBPV



- Adapted from: Forster, Kunze, and Roth (2020).
- Mechanised in this thesis

FIGURE 4.10: Simulation between Turing Machines and CBPV with intermediate models

Thank you

Questions?

Formalisation available at:

https://github.com/ZhuoZoeyChen/cbpv-reasonable-HOL/

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Method - Why CBPV?

Evaluation Strategies for λ-Calculus

evaluates first

Evaluates first

call by value

fst(1+2, 1+3)
= fst(3, 1+3)
= fst(3, 4)
= 3
Evaluates the function argument before passing them into the function call

substitutes first

Evaluates after

call by name

fst(1+2, 1+3)
= 1+2
= 3
Copying the function argument into the function body without evaluating it first

lazy

Evaluates when needed

call by need

fst(1+2, 1+3)
1. let x = 1+2
    y = 1+3
  in fst(x, y)
2. let x = 1+2
    y = 1+3
  in x
3. let x = 3 in x
4. 3

weak

reduction doesn't go under abstraction

strong

reduction does go under abstraction

Method - Why CBPV?

fst(1+2, 1+3)
= fst(3, 1+3)
= fst(3, 4)
= 3

call by value

fst(1+2, 1+3)
= 1+2
= 3

call by name

fst(1+2, 1+3)
1. let x = 1+2
   y = 1+3
   in fst(x, y)
2. let x = 1+2
   y = 1+3
   in x
3. let x = 3 in x
4. 3

call by need

Evaluation Strategies for λ-Calculus

weak

reduction doesn't go under abstraction

strong