Barendregt's Theory of the λ-Calculus, Refreshed and Formalized

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Outline

Barendregt's Theory of the Lambda Calculus

Formalizing in Abella

Partial Recursive Functions

Partial Recursive Functions model which mathematical functions are computable.

There is a natural *extensional preorder* on partial functions

$$f \leq_{\mathrm{PRF}} g \text{ if } \forall n \in \mathbb{N},\ f(n) = \bot \text{ or } f(n) =_{\mathbb{N}} g(n)$$

$f_\bot : n \mapsto \bot$ is the minimum PRF function for $\leq_{\mathrm{PRF}}$

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Lambda Calculus

PRF do not look at how to compute, hence the preorder can only be extensional.

Instead, in the lambda calculus, how to compute is a critical concept.

There are a rich number of possible equivalences (or preorders) of lambda terms, both extensional or intensional.

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Computable Functions & Lambda Calculus

Partial recursive functions embed in the lambda calculus.

What is the lambda term that represents **undefined**?

A computation that never ends? Ω!

$\Omega := (\lambda x.xx)(\lambda x.xx) \rightarrow_\beta (\lambda x.xx)(\lambda x.xx) \rightarrow_\beta \cdots$

Now, what is the equivalence class of **undefined**/Ω ?

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A first naive attempt

Undefined represents a computation that never ends.

- **undefined** terms = β-diverging terms?

Surprisingly, this would lead to an inconsistency.

>> If all β-diverging terms are equated in an equational theory, then this theory equates all terms.

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β-diverging terms may be very different

Indeed, let us look at two β-diverging terms

$$\begin{array}{ccc}
\mathit{fix} & \text{and} & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\lambda f.f\ (\mathit{fix}\, f) & & \Omega \\
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\lambda f.f\ (f\ (f\ (f\ \ldots))) & & \Omega \\
\downarrow_\beta & & \downarrow_\beta \\
\vdots & & \vdots
\end{array}$$

Recursion does not carry the same meaning as looping on itself.

A second attempt

Instead, one might consider a more restrained reduction

- **undefined** terms $=$ **head**-diverging terms?

The equational theory that identifies **head**-diverging terms is consistent.

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Fixpoint combinators are **head**-normalizing.

$$\begin{array}{ccc}
\mathit{fix} & \text{and} & \Omega \\
\downarrow_h & & \downarrow_h \\
\lambda f.f\ (\mathit{fix}\, f) & & \Omega \\
\not\downarrow_h & & \downarrow_h \\
 & & \vdots
\end{array}$$

Recursion and looping are nicely separated by **head** reduction.

Consistency

A relation $\mathcal{R} \subseteq \Lambda \times \Lambda$ is consistent if there exists $t, u \in \Lambda$ such that $(t, u) \notin \mathcal{R}$.

An equational theory is an equivalence relation $=_{\mathcal{T}}$ such that:

- *Invariance under Computation*: if $t \to_\beta u$ then $t =_{\mathcal{T}} u$
- *Stability by Contexts*: if $t =_{\mathcal{T}} u$ then $\forall C$, $C\langle t\rangle =_{\mathcal{T}} C\langle u\rangle$.

To validate the choice of undefined terms: Is there a consistent equational theory where undefined terms are collapsed?

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What is Genericity?

Undefined terms are black holes for the evaluation process.

$$u \longrightarrow u_1 \xrightarrow{n} u_n \xrightarrow{*} \cdots$$

If a program awaits the evaluation of an undefined sub-term

$$\cdots \longrightarrow \boxed{C\; u} \longrightarrow \boxed{C\; u_1}$$

Then it will be unable to produce a result

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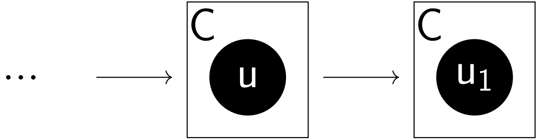

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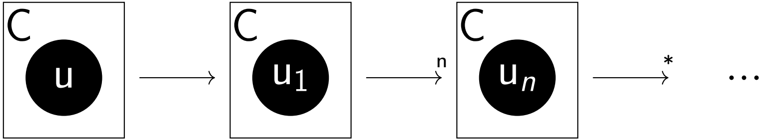

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Genericity somehow specifies this fact dually:
If a program terminates while there were **undefined** sub-terms, then it never entered the black hole.

Genericity says: (n is a normal form and s is any term)

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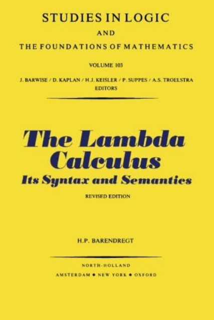

Refreshed:

- Takahashi's proof of genericity (1994)
- Accattoli et al. study of normalization (2019)

Formalized with the Abella proof assistant:

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- Representing contexts (with possible captures)

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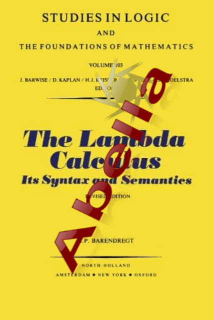

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Proving/Formalizing Genericity

A Simple Proof of the Genericity Lemma

Masako Takahashi

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Abstract. A short direct proof is given for the fundamental property of unsolvable λ-terms; if M is an unsolvable λ-term and $C[M]$ is solvable, then $C[N]$ is solvable for any λ-term N. (Here $C[\ ]$ stands for an arbitrary context.)

1. Preliminaries

A term in this note means a λ-term, which is either $x, \lambda x.M$ or MN, (where M, N are terms and x is a variable.) Unless otherwise stated, capital letters $M, N, P, \ldots$ stand for arbitrary terms, $\mathbf{M}, \mathbf{N}, \ldots$ for (possibly null) sequences of terms, $x, y, \ldots$ for variables, and $\mathbf{x}, \mathbf{y}, \ldots$ for (possibly null) sequences of variables. We refer to [1] as the standard text in the field.

A term of the form $\lambda \mathbf{x}.y\mathbf{M}$ (more precisely, $\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.((\ldots((yM_1)M_2)\ldots)M_m))\ldots))$ for some $n, m \geq 0$) is said to be in *head normal form* (*hnf*, for short). If a term M has a hnf (that is, $M =_\beta M'$ for a term M' in hnf), then M is called *solvable*. The following are well-known facts of solvable terms (cf.[1] 8.3.1 - 14).

(1) M is solvable if and only if $\forall P, \exists \mathbf{x}, \exists \mathbf{Q}((\lambda \mathbf{x}.M)\mathbf{Q} =_\beta P)$.
(2) $\lambda x.M$ is solvable if and only if so is M.
(3) if $M[x := N]$ is solvable then so is M.
(4) if MN is solvable then so is M.

A term in *β-normal form* (*β-nf*, for short) is recursively defined as a term of the form $\lambda \mathbf{x}.y\mathbf{M}$ where $\mathbf{M}$ is a (possibly null) sequence of terms in β-nf.

2. Propositions

First we prove a special case of the genericity lemma.

Lemma 1. Let M, N, P be terms with M unsolvable and N in β-nf. Then $P[x := M] =_\beta N$ implies $P[x := M'] =_\beta N$ for any M'.

Proof. We prove the lemma by induction on the structure of N. Suppose $P[x := M] =_\beta N$, and $N \equiv \lambda \mathbf{y}.zN_1N_2\ldots N_n$ where $n \geq 0$ and each N_i is in β-nf. (Here, $\equiv$ denotes syntactic equality of terms.) Then since N is solvable, P is also solvable by (3) above, and hence has a hnf, say $\lambda \mathbf{u}.vP_1P_2\ldots P_p$. Here x and v must be different. (For otherwise $P[x := M] =_\beta \lambda \mathbf{u}.M\mathbf{P}$ for some $\mathbf{P}$, and $P[x := M]$ would by (2) and (4) above be unsolvable, which contradicts our assumption.) Therefore we have $P[x := M] =_\beta \lambda \mathbf{u}.vP'_1P'_2\ldots P'_p$ where $P'_i \equiv P_i[x := M](i = 1, 2, \ldots, p)$. Since $P[x := M] =_\beta N \equiv \lambda \mathbf{y}.zN_1N_2\ldots N_n$, we know from the Church-Rosser theorem that $P'_i =_\beta N_i(i = 1, 2, \ldots, n)$ and $p = n$. Without loss of generality we may also assume $\mathbf{u} \equiv \mathbf{y}$ and $v \equiv z$.

If $n = 0$, then $P =_\beta \lambda \mathbf{u}.v \equiv N$. In this case, we have $P[x := M'] =_\beta (\lambda \mathbf{u}.v)[x := M'] \equiv \lambda \mathbf{u}.v \equiv N$ for any M'. If $n > 0$, then from the fact $P_i[x := M] \equiv P'_i =_\beta N_i$ and the inductive hypothesis, we get $P_i[x := M'] =_\beta N_i(i = 1, 2, \ldots, n)$ for any M'. In this case,

$$\begin{aligned} P[x := M'] &=_\beta (\lambda \mathbf{u}.vP_1P_2\ldots P_n)[x := M'] \\ &\equiv \lambda \mathbf{u}.v(P_1[x := M'])(P_2[x := M'])\ldots(P_n[x := M']) \\ &=_\beta \lambda \mathbf{y}.zN_1N_2\ldots N_n \equiv N. \end{aligned}$$

This proves the lemma. □

Lemma 2. ([1] 14.3.24. Genericity lemma) Let M be an unsolvable term, and $C[\ ]$ be a context such that $C[M]$ has a β-nf. Then $C[M] =_\beta C[M']$ for any M'.

Proof. For given M', let $\mathbf{y}$ be a sequence of all free variables in MM'. Take a new variable x (neither in $C[M]$ nor $C[M']$), and let $P \equiv C[x\mathbf{y}]$. Then since $\lambda \mathbf{y}.M$ and $\lambda \mathbf{y}.M'$ are closed terms, we have

$$P[x := \lambda \mathbf{y}.M] \equiv C[(\lambda \mathbf{y}.M)\mathbf{y}] =_\beta C[M],$$
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The term $\lambda \mathbf{y}.M$ therefore satisfies $P[x := \lambda \mathbf{y}.M] =_\beta C[M] =_\beta N$ for some N in β-nf. Here $\lambda \mathbf{y}.M$ is unsolvable because so is M. Hence by applying lemma 1 we get $P[x := \lambda \mathbf{y}.M'] =_\beta N$, which implies $C[M] =_\beta C[M']$. □

Corollary 3. If M is unsolvable and $C[M]$ is solvable, then $C[M']$ is solvable for any M'.

Proof. Since $C[M]$ is solvable, by (1) above there exist $\mathbf{x}$ and $\mathbf{N}$ such that $(\lambda \mathbf{x}.C[M])\mathbf{N}$ has a β-nf. Then by lemma 2 (applied to the context $(\lambda \mathbf{x}.C[\ ])\mathbf{N}$), we know $(\lambda \mathbf{x}.C[M'])\mathbf{N}$ has a β-nf for any M'. This means $(\lambda \mathbf{x}.C[M'])\mathbf{N}$ is solvable, and consequently $C[M']$ is solvable. □

The proof presented provides an alternative to the conventional one which uses a topological argument on Böhm trees (cf.[1] Chapters 10 and 14).

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Main Lemma
~15 lines of text
~90 lines of Abella
~140 tactics

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If $n = 0$, then $P =_\beta \lambda\mathbf{u}.v \equiv N$. In this case, we have $P[x := M'] =_\beta (\lambda\mathbf{u}.v)[x := M'] \equiv \lambda\mathbf{u}.v \equiv N$ for any M'. If $n > 0$, then from the fact $P_i[x := M] \equiv P'_i =_\beta N_i$ and the inductive hypothesis, we get $P_i[x := M'] =_\beta N_i(i = 1, 2, ..., n)$ for any M'. In this case,

$$\begin{aligned} P[x := M'] &=_\beta (\lambda\mathbf{u}.vP_1P_2...P_n)[x := M'] \\ &\equiv \lambda\mathbf{u}.v(P_1[x := M'])(P_2[x := M'])...(P_n[x := M']) \\ &=_\beta \lambda\mathbf{y}.zN_1N_2...N_n \equiv N. \end{aligned}$$

This proves the lemma. □

Lemma 2. ([1] 14.3.24. Genericity lemma) Let M be an unsolvable term, and $C[\ ]$ be a context such that $C[M]$ has a β-nf. Then $C[M] =_\beta C[M']$ for any M'.
Proof. For given M', let $\mathbf{y}$ be a sequence of all free variables in MM'. Take a new variable x (neither in $C[M]$ nor $C[M']$), and let $P \equiv C[x\mathbf{y}]$. Then since $\lambda\mathbf{y}.M$ and $\lambda\mathbf{y}.M'$ are closed terms, we have

$$P[x := \lambda\mathbf{y}.M] \equiv C[(\lambda\mathbf{y}.M)\mathbf{y}] =_\beta C[M],$$
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The term $\lambda\mathbf{y}.M$ therefore satisfies $P[x := \lambda\mathbf{y}.M] =_\beta C[M] =_\beta N$ for some N in β-nf. Here $\lambda\mathbf{y}.M$ is unsolvable because so is M. Hence by applying lemma 1 we get $P[x := \lambda\mathbf{y}.M'] =_\beta N$, which implies $C[M] =_\beta C[M']$. □

Corollary 3. If M is unsolvable and $C[M]$ is solvable, then $C[M']$ is solvable for any M'.
Proof. Since $C[M]$ is solvable, by (1) above there exist $\mathbf{x}$ and $\mathbf{N}$ such that $(\lambda\mathbf{x}.C[M])\mathbf{N}$ has a β-nf. Then by lemma 2 (applied to the context $(\lambda\mathbf{x}.C[\ ])\mathbf{N}$), we know $(\lambda\mathbf{x}.C[M'])\mathbf{N}$ has a β-nf for any M'. This means $(\lambda\mathbf{x}.C[M'])\mathbf{N}$ is solvable, and consequently $C[M']$ is solvable. □

The proof presented provides an alternative to the conventional one which uses a topological argument on Böhm trees (cf.[1] Chapters 10 and 14).

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Preliminaries:
~2000 lines of Abella

Main Lemma
~15 lines of text
~90 lines of Abella
~140 tactics

Proving/Formalizing Genericity

A Simple Proof of the Genericity Lemma

Takahashi's trick
(**Disentangling**)
~60 lines of Abella

Abstract. A short direct p
is an unsolvable λ-term and
stands for an arbitrary cont

1. Preliminaries

A term in this note means a λ-term, which is either $x, \lambda x.M$ or MN, (where M, N are terms and x is a variable.) Unless otherwise stated, capital letters $M, N, P, \ldots$ stand for arbitrary terms, $\mathbf{M}, \mathbf{N}, \ldots$ for (possibly null) sequences of terms, $x, y, \ldots$ for variables, and $\mathbf{x}, \mathbf{y}, \ldots$ for (possibly null) sequences of variables. We refer to [1] as the standard text in the field.

A term of the form $\lambda\mathbf{x}.y\mathbf{M}$ (more precisely, $\lambda x_1.(\lambda x_2.(\ldots(\lambda x_n.((\ldots((yM_1)M_2)\ldots)M_m))\ldots))$ for some $n, m \geq 0$) is said to be in *head normal form* (*hnf*, for short). If a term M has a hnf (that is, $M =_\beta M'$ for a term M' in hnf), then M is called *solvable*. The following are well-known facts of solvable terms (cf.[1] 8.3.1 - 14).

(1) M is solvable if and only if $\forall P, \exists \mathbf{x}, \exists \mathbf{Q}((\lambda\mathbf{x}.M)\mathbf{Q} =_\beta P)$.
(2) $\lambda x.M$ is solvable if and only if so is M.
(3) if $M[x := N]$ is solvable then so is M.
(4) if MN is solvable then so is M.

A term in *β-normal form* (*β-nf*, for short) is recursively defined as a term of the form $\lambda\mathbf{x}.y\mathbf{M}$ where $\mathbf{M}$ is a (possibly null) sequence of terms in β-nf.

2. Propositions

First we prove a special case of the genericity lemma.

Lemma 1. Let M, N, P be terms with M unsolvable and N in β-nf. Then $P[x := M] =_\beta N$ implies $P[x := M'] =_\beta N$ for any M'.

Proof. We prove the lemma by induction on the structure of N. Suppose $P[x := M] =_\beta N$, and $N \equiv \lambda\mathbf{y}.zN_1N_2\ldots N_n$ where $n \geq 0$ and each N_i is in β-nf. (Here, $\equiv$ denotes syntactic equality of terms.) Then since N is solvable, P is also solvable by (3) above, and hence has a hnf, say $\lambda\mathbf{u}.vP_1P_2\ldots P_p$. Here x and v must be different. (For otherwise $P[x := M] =_\beta \lambda\mathbf{u}.M\mathbf{P}$ for some $\mathbf{P}$, and $P[x := M]$ would by (2) and (4) above be unsolvable, which contradicts our assumption.) Therefore we have $P[x := M] =_\beta \lambda\mathbf{u}.vP_1'P_2'\ldots P_p'$ where $P_i' \equiv P_i[x := M](i = 1, 2, \ldots, p)$. Since $P[x := M] =_\beta N \equiv \lambda\mathbf{y}.zN_1N_2\ldots N_n$, we know from the Church-Rosser theorem that $P_i' =_\beta N_i(i = 1, 2, \ldots, n)$ and $p = n$. Without loss of generality we may also assume $\mathbf{u} \equiv \mathbf{y}$ and $v \equiv z$.

If $n = 0$, then $P =_\beta \lambda\mathbf{u}.v \equiv N$. In this case, we have $P[x := M'] =_\beta (\lambda\mathbf{u}.v)[x := M'] \equiv \lambda\mathbf{u}.v \equiv N$ for any M'. If $n > 0$, then from the fact $P_i[x := M] \equiv P_i' =_\beta N_i$ and the inductive hypothesis, we get $P_i[x := M'] =_\beta N_i(i = 1, 2, \ldots, n)$ for any M'. In this case,

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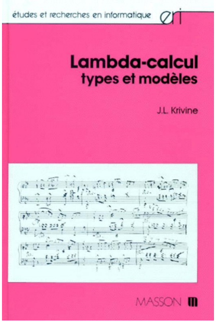
Some Related Work

Many other formal developments of the theory of the λ-calculus



$\leftarrow$ **Formalization** of parts of Krivine's book (1990) in Rocq by Larchey-Wendling

- Countless formalized proofs of confluence
- The previous talk!
- ...

Contextual Preorder

The **head (open) contextual preorder** is defined as:

$t \precsim^{\mathtt{h}}_{\mathcal{CO}} u$ if **for all contexts** C, $C\langle t\rangle$ is $\mathtt{h}$-normalizing implies $C\langle u\rangle$ is $\mathtt{h}$-normalizing.

- A natural extensional inequational theory
  The only non-trivial point is the inclusion of β-conversion.
- Strongly Connected with Genericity:
  Genericity says that "head diverging terms are **minimums** for the head contextual preorder".

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Outline

Formalizing λ

$$\textsc{Terms} \quad t, u \quad := \quad x \mid \lambda x.t \mid tu$$

λ-terms and the predicate for inducting on them in Abella:

```
Kind  tm   type.

Type  abs  (tm -> tm) -> tm.
Type  app  tm -> tm -> tm.


Define is_tm : tm -> prop by
  nabla x, is_tm x;
  is_tm (abs T) := nabla x, is_tm (T x);
  is_tm (app T U) := is_tm T /\ is_tm U.
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Formalizing λ and β

$$\frac{}{(\lambda x.t)u \mapsto_\beta t\{x \leftarrow u\}} \quad \frac{t \to_\beta t'}{tu \to_\beta t'u} \quad \frac{t \to_\beta t'}{\lambda x.t \to_\beta \lambda x.t'} \quad \frac{u \to_\beta u'}{tu \to_\beta tu'}$$

```
Define beta : tm -> tm -> prop by
  beta (app (abs T) U) (T U);
  beta (app T U) (app T' U) := beta T T';
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Formalizing λ-theories

A λ-theory is stable by contexts...

Contextual equivalence...

We need to formalize contexts

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Formalizing λ-theories

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Formalizing Contexts

A context is a term with a hole? Not really...

Set $C := \lambda x.\langle\cdot\rangle$, then $C\langle y\rangle = \lambda x.y$ and $C\langle x\rangle = \lambda x.x = \mathtt{I}$.

`ctx T CT` holds iff there exists a context C such that $C\langle \mathtt{T}\rangle = \mathtt{CT}$.

```
Define ctx : tm -> tm -> prop by
  ctx T T;
  ctx T (app P Q) := ctx T P \/ ctx T Q;
  nabla x, ctx (T x) (abs CT) :=
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```

How to apply a context to two different terms?

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`ctxs T CT U CU` holds
iff there exists a context C such that $C\langle \texttt{T}\rangle = \texttt{CT}$ and $C\langle \texttt{U}\rangle = \texttt{CU}$.

```
Define ctxs : tm -> tm -> tm -> tm -> prop by
  ctxs T T U U;
  ctxs T (app A B) U (app C D) :=
    (ctxs T A U C /\ B = D /\ tm D)
    \/ (ctxs T B U D /\ A = C /\ tm C);
  nabla x, ctxs (T x) (abs CT) (U x) (abs CU):=
    nabla y, ctxs (T y) (CT y) (U y) (CU y).
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Formalizing Contextual Preorder

```
Define ctx_preord : tm -> tm -> prop by
  ctx_preord P Q := forall CP CQ,
    tm P -> tm Q ->
    ctxs P CP Q CQ -> head_terminating CP ->
    head_terminating CQ.
```

- `ctx_preord` is stable by contexts.
- `ctx_preord` is invariant under computation.
- `ctx_preord` has `h`-diverging terms as minimums.

Light Genericity

Light Genericity: head-diverging terms are minimum for the head open contextual preorder.

Unfolded statement:

Light Genericity: let u be head-diverging and C such that $C\langle u\rangle$ is head-normalizing then $C\langle t\rangle$ is head-normalizing for all $t \in \Lambda$.

Main difficulty: reasoning with contexts and reduction.

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Direct proof of Light Genericity

Takahashi proves Barendregt's heavy genericity with a very short proof [Tak94] and gives as a corollary light genericity.

Key idea/trick: Reason with substitutions instead of contexts!

Light genericity as substitution: let u be $\mathtt{h}$-diverging and t such that $t\{x{\leftarrow}u\}$ is $\mathtt{h}$-normalizing then $t\{x{\leftarrow}s\}$ is $\mathtt{h}$-normalizing for all $s \in \Lambda$.

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Takahashi's Trick in CbN

$$\begin{array}{rcl} C\langle u\rangle & \leftrightarrow & t_C\{x\leftarrow u_C\} \\ C\langle s\rangle & \leftrightarrow & t_C\{x\leftarrow s_C\} \end{array}$$

Trick:
Let $\mathtt{fv}(u) \cup \mathtt{fv}(s) = \{x_1, \ldots, x_k\}$, and y a fresh variable.

- $u_C := \lambda x_1. \ldots . \lambda x_k.u$ and $s_C := \lambda x_1. \ldots . \lambda x_k.s$ are closed terms.
- Consider $t_C := C\langle yx_1 \ldots x_k\rangle$, and note that:

$$\begin{array}{rcl} t_C\{y\leftarrow u_C\} & = & C\langle u_C x_1 \ldots x_k\rangle \\ & = & C\langle(\lambda x_1. \ldots . \lambda x_k.u)x_1 \ldots x_k\rangle \\ & \to_\beta^k & C\langle u\rangle \end{array}$$

- u is h-diverging implies that u_C is also h-diverging.
- $C\langle u\rangle$ is h-normalizing if and only if $t\{y\leftarrow u_C\}$ is. (also true for s and s_C)

  by the Head Normalization Theorem (and confluence, etc.)

Formalizing Takahashi's Trick

$$C\langle u\rangle \;\leftrightarrow t_C\{x{\leftarrow}u_C\}$$

Disentangling:
For any context C, there exist t_C and a variable $x \notin \mathtt{fv}(C)$ such that:

- for all terms u there exists u_C such that $t_C\{x{\leftarrow}u_C\} \to_\beta^* C\langle u\rangle$. (Moreover, if u is head divergent then u_C is head divergent.)

Some small technicalities in Abella...

Substitution Preorder:

$u \precsim_{\mathcal{S}}^{\mathrm{h}} s$ holds if
for all terms t, variables x, and lists of variables $y_1, \ldots, y_n$ with $n \geq 0$,
$t\{x{\leftarrow}\lambda y_1.\ldots.\lambda y_n.u\}$ $\to_{\mathrm{h}}$-terminating implies that $t\{x{\leftarrow}\lambda y_1.\ldots.\lambda y_n.s\}$ is $\to_{\mathrm{h}}$-terminating

The substitution preorder coincides with the contextual preorder.

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Maximality

Another result in Barendregt's book:

Maximality of the Head Contextual Preorder:
if $\precsim^{\mathtt{h}}_{\mathcal{CO}} \subsetneq \leq_{\mathcal{T}}$ then $\leq_{\mathcal{T}}$ is inconsistent.

$\ll$ The head contextual preorder is the largest sensible theory to study. $\gg$

Constructive Contextual Equivalence?

Proofs of maximality always starts by:

If $\mathcal{T} \vdash t \leq u$ and $t \not\lesssim^{\mathtt{h}}_{\mathcal{CO}} u$
Then $\underline{\exists C}$ such that

- $C\langle t\rangle$ is $\mathtt{h}$-normalizing and
- $C\langle u\rangle$ is $\mathtt{h}$-diverging.

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In general, not valid in intuistionistic logic

$$\neg\forall\phi \not\Rightarrow \exists\neg\phi$$

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- Easy proofs that rely mostly on rewriting/operational results
- Faithful formalization of the pen-and-paper proofs

Future work:

- Constructive Contextual (In)Equivalence?
- Many results adapt to the theory of the Call-by-Value calculus (haven't formalized these)
- Other results on program equivalence to be made formal (mechanizing Böhm trees and Böhm 's theorem?
  $\Rightarrow$ intensional presentation of contextual equivalence)

Thank you!

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Thank you!

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