Verifying an Efficient Algorithm to Compute Bernoulli Numbers

Manuel Eberl Peter Lammich

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$$\sum_{i=0}^{n} i^k = \tfrac{1}{m+1}(P_{m+1}(n+1) - P_{m+1}(0)) \qquad \text{where } P_m(x) = \sum_{0 \le k \le m} \binom{m}{k} B_k x^{m-k}$$

Bernoulli Numbers

Definition

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The Bernoulli numbers are the sequence of rationals $(B_k)_{k\geq 0}$ with

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k	0	1	2	3	4	5	6	7	8	9	10
B_k	1	$-\frac{1}{2}$	$\frac{1}{6}$	0	$-\frac{1}{30}$	0	$\frac{1}{42}$	0	$-\frac{1}{30}$	0	$\frac{5}{66}$

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$B_{60} \approx -1.2 \cdot 10^{42} \;/\; 56786730$ $\qquad$ $B_{110} \approx 7.6 \cdot 10^{93} \;/\; 1518$

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Theorem (Voronoi's congruence)

$$(a^k - 1)B_k \equiv ka^{k-1} \sum_{1 \le m < p} m^{k-1} \left\lfloor \frac{ma}{p} \right\rfloor \pmod{p}$$

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Theorem (Kummer's congruence)

$$B_k/k \equiv B_{k'}/k \pmod{p} \qquad \text{where } k' = k \bmod (p-1)$$

Thus we can w.l.o.g. assume that $k = 2, 4, \ldots, p-3$.

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  Combine results via *Chinese Remainder Theorem*.

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- Read off N_k

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Size of blocks chosen so that table fits into L1d cache.

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Example applications:

- IsaSAT by Fleury: fully verified SAT solver
- Lammich: verified sorting algorithms on par with C++ standard library

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lemma "(factor_cache_ll_impl, factor_cache_impl2)
  ∈ unat_assn^k $\rightarrow_a$ array_assn unat_assn $\times_a$ unat_assn"

Final correctness theorem

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Challenges

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- Concrete bounds for the Chebyshev ϑ function:

$$\vartheta(x) = \sum_{p \le x} \ln p \ge 0.82x \quad \text{for } x \ge 97$$

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- Bounds for Bernoulli numbers Easy.
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Abstract algorithm	1500
Concrete algorithm	8500
Total	31700

Low-level optimisations

We can compute the binary fraction expansion of $1/p$ in 64-bit chunks by letting `bitbuf = 1` and then repeating

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output ((bitbuf << 64) / p)
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- remainder `(bitbuf << 64) % p` via `-p * bitbuf_new`.

There is a small chance that the result is off-by-one, which we have to detect and correct accordingly.

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```
// bitbuf := 2 ^ 64 * x / p,   x := 2 ^ 64 * x % p
uint64_t bitbuf = (invp * (__uint128_t) x) >> 64;
x = -p * bitbuf;

// There is a small chance that our quotient is actually too small by 1; we detect this here.
if (__builtin_expect(x >= p, 0)) [[unlikely]] {
    bitbuf++;
    x -= p;
}
```

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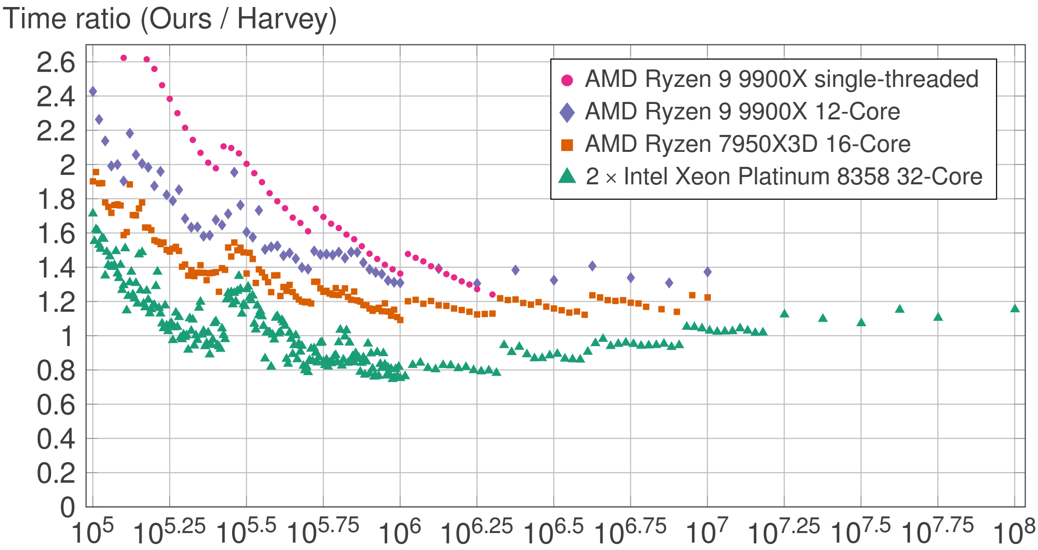
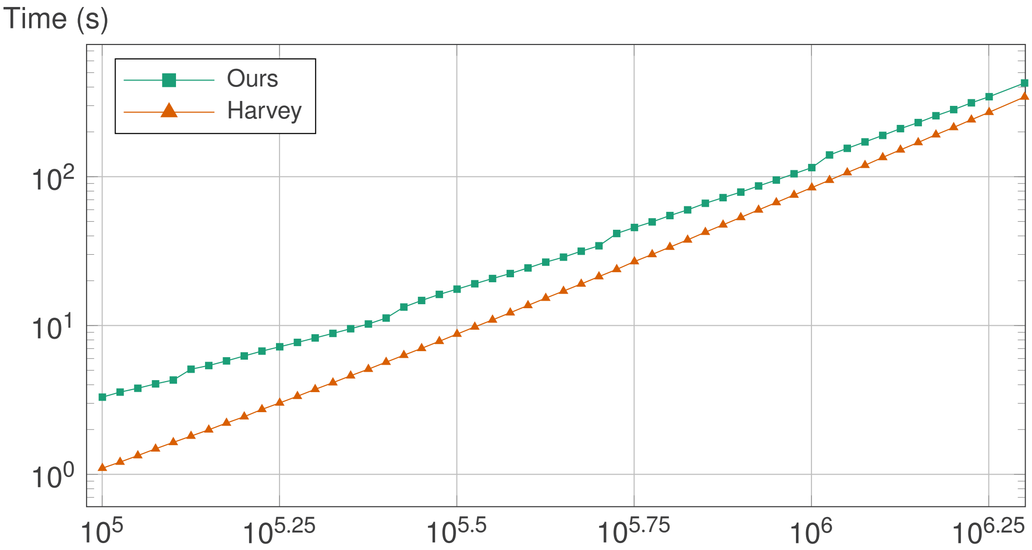
Lessons learnt:

- Proving absence of overflow can be painful.
- Advantage: Using an ITP helps you figure out the range in which the algorithm does not produce overflow.

Evaluation



universität innsbruck UNIVERSITY OF TWENTE.


Time (s)
Ours
Harvey
10^2
10^1
10^0
10^5
$10^{5.25}$
$10^{5.5}$
$10^{5.75}$
10^6
$10^{6.25}$

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We are already *much* faster than Mathematica's `BernoulliB` algorithm!