Formalizing Possibly Infinite Trees of Finite Degree

König's Lemma for Chase Termination

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Background / Short Motivation 💡

My work revolves around the chase algorithm on existential rules.

See https://dmfa.dev/lean

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With the fact $R(a, b)$, we can obtain different `ChaseBranches`.

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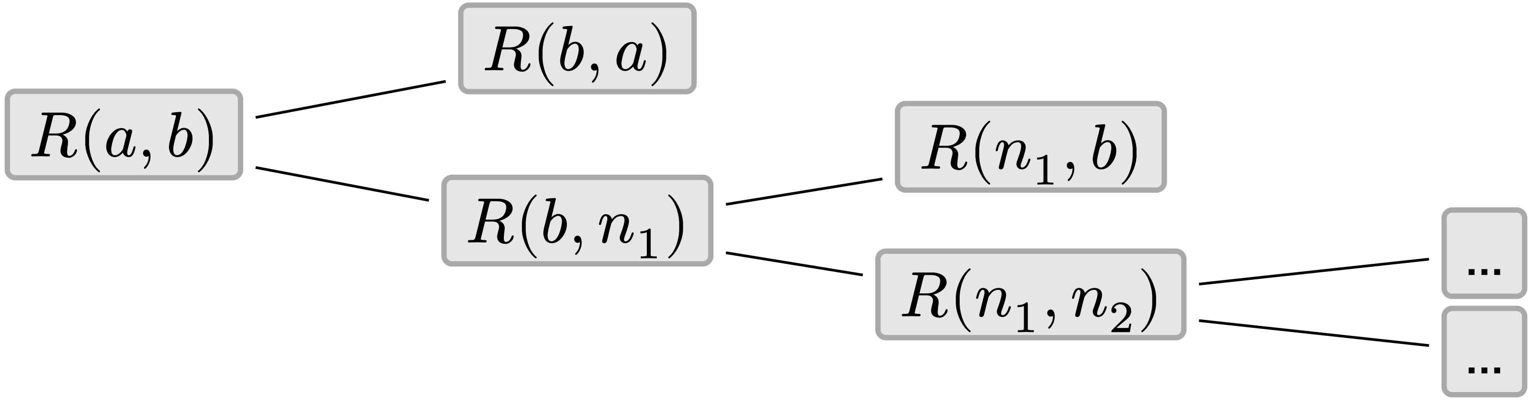
$$\boxed{R(x, y) \to \exists z.R(y, z) \lor R(y, x)}$$

We can generalize this into an infinite `ChaseTree`.



What about inductive trees? 🌲

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Inductive types express the least fixed point, i.e. the minimal set of trees obtainable from the two constructors.

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Inductive types express the least fixed point, i.e. the minimal set of trees obtainable from the two constructors. We want the greatest fixed point instead, i.e. a coinductive definition!

Plan of Attack ⚔️

1.
2.
3.
4.
5.

https://dmfa.dev/trees

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1. Take a hint from infinite lists.
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Plan of Attack ⚔️

1. Take a hint from infinite lists.
2. Define possibly infinite lists.
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1. Take a hint from infinite lists.
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3. Define possibly infinite trees with finite degree.
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4. Define branches in trees.
5.

https://dmfa.dev/trees

Plan of Attack ⚔️

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3. Define possibly infinite trees with finite degree.
4. Define branches in trees.
5. Prove König's Lemma.

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König's Lemma: (Special Case for Trees)

If each branch of a tree with finite degree is finite,

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König's Lemma: (Special Case for Trees)
If each branch of a tree with finite degree is finite,
then there are only finitely many branches.

Infinite Lists aka. Stream' 🏄

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structure PossiblyInfiniteList (α : Type u) where
  infinite_list : InfiniteList (Option α)
  no_holes : ∀ n : Nat, infinite_list n ≠ none ->
             ∀ m : Fin n, infinite_list m ≠ none
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Now we only need to turn the `InfiniteList` into an `InfiniteTree`. How?

From List to Tree - Change the Address!

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-- List addresses are `Nat`. Tree addresses are `List Nat`!
def FiniteDegreeTreeStub (a : Type u) := List Nat -> Option a
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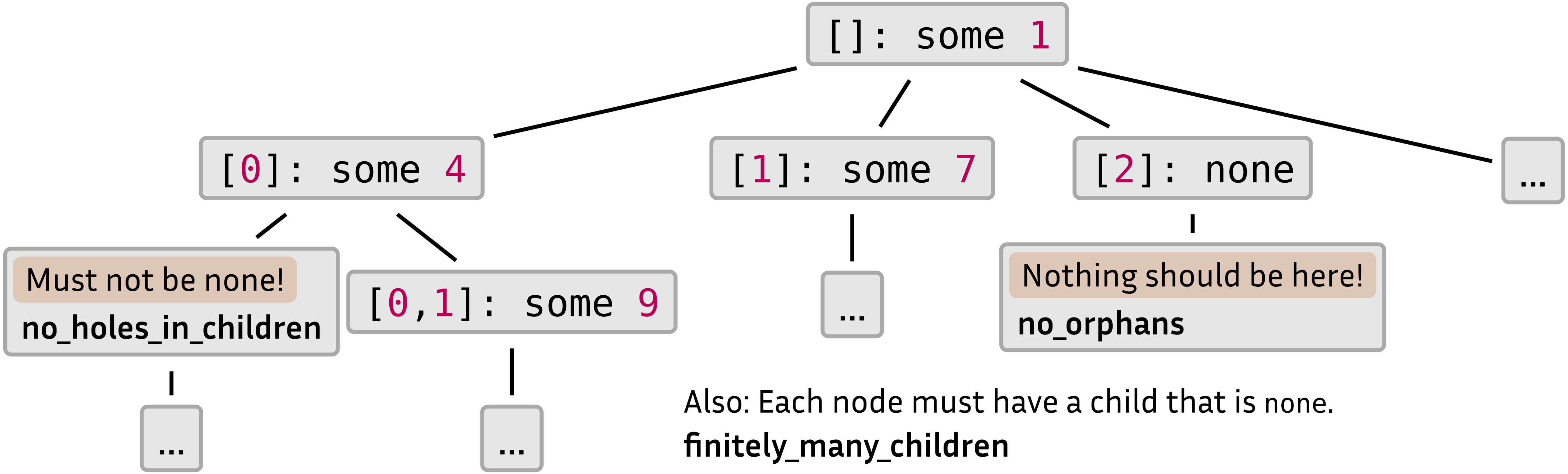

Also: Each node must have a child that is none.
finitely_many_children

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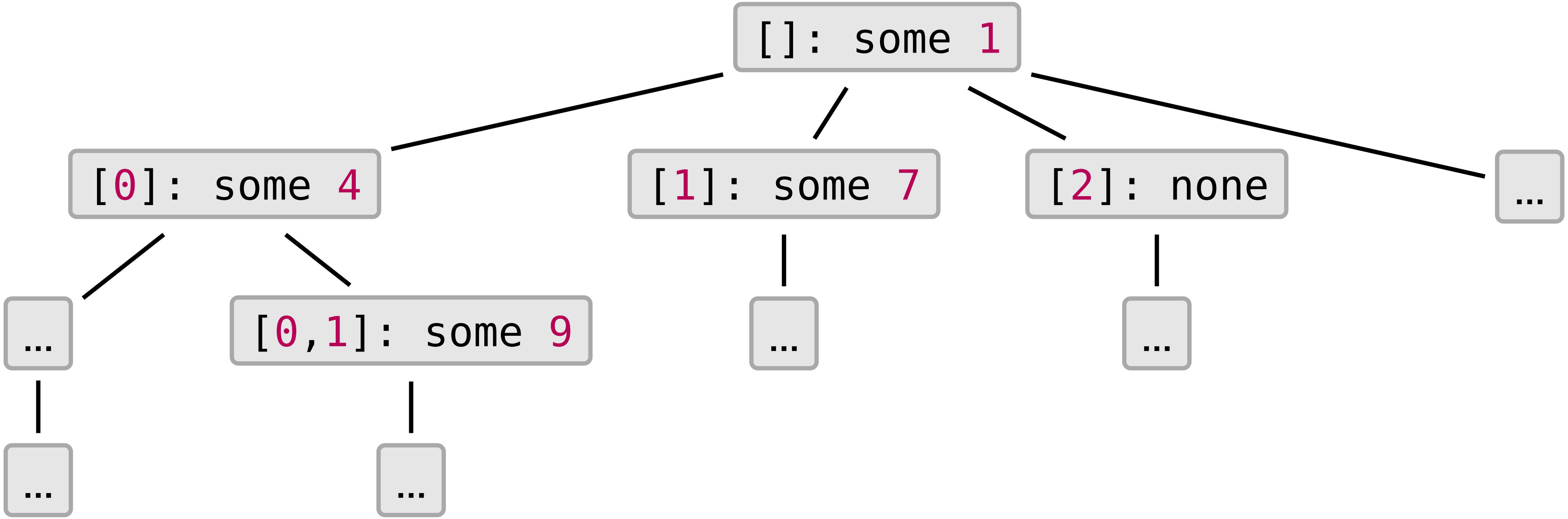
Is that it?

Defining Branches 🌳 (2)



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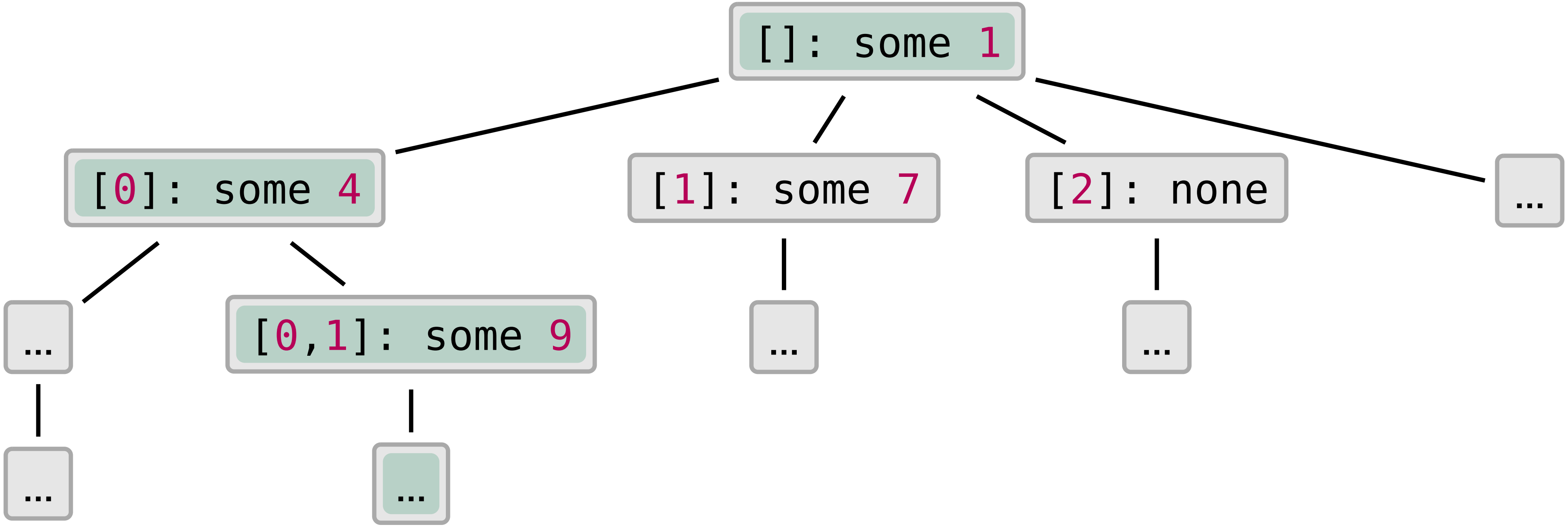
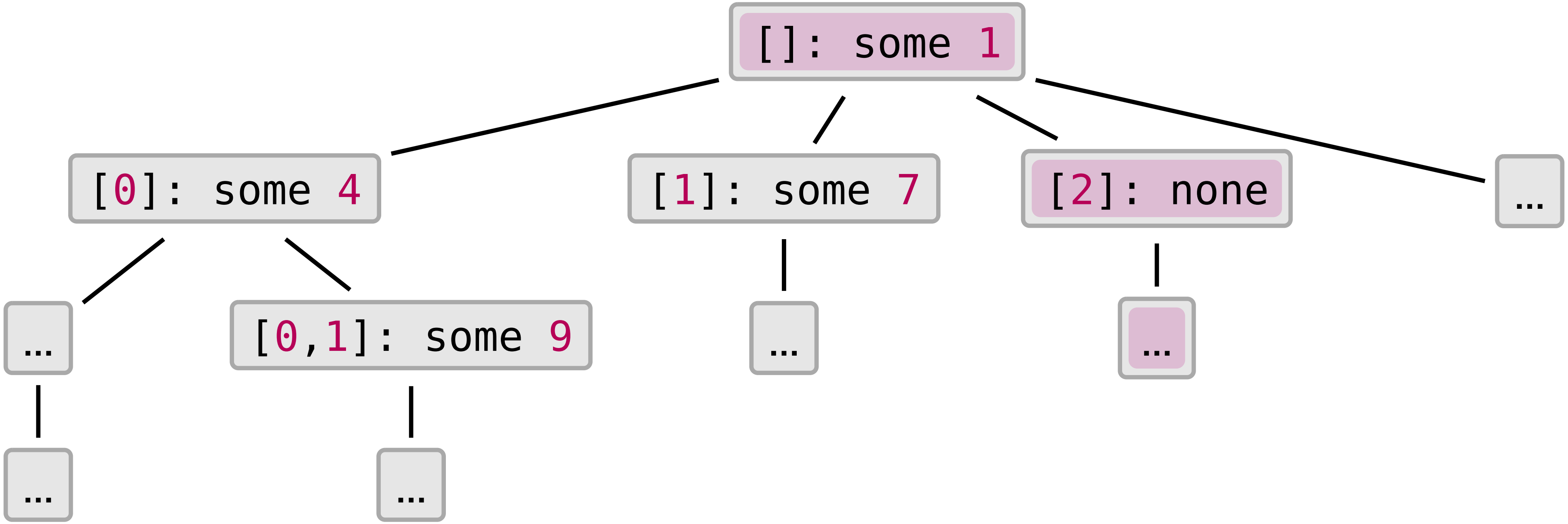


Should the green be a valid branch address? ✋

Defining Branches 🥀 (2)



Should the pink be a valid branch address? ✋

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def address_is_maximal (t : PIT α) (ns : IL Nat) : Prop :=
  -- If the branch is not infinite, it should end in a leaf
  ∀ n, t (ns.take (n+1)) = none -> t (0 :: (ns.take n)) = none
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📢 This also ensures that every `ChaseTree` branch is a proper `ChaseBranch`.

Proving König's Lemma 👑

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If each branch of a tree with finite degree is finite,
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Proving König's Lemma 👑

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theorem branches_finite_of_each_branch_finite (t : FinDegTree α) :
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have : ∃ (ns : IL Nat), ∀ i, ¬ t.branches_through (ns.take i).fin
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theorem inf_node_extends_prev {t} (n_fin : ¬ t.branches.fin) (d) :
  (inf_node n_fin d.succ).val =
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📢 If every branch in a `ChaseTree` is finite, so is the whole `ChaseTree`.

This talk is finite and we've reached its end 🏁

Thank you so much for having me! 🤓

I hope you can get all your `goals accomplished` 🎉

Feel free to reach out in Lean Zulipchat or via mail:

`lukas.gerlach@tu-dresden.de`

`hi@monsterkrampe.dev`

⭐ Check out https://dmfa.dev/lean and https://dmfa.dev/trees ⭐