Euler's polyhedron formula

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Lean @ ITP 2025
2025-10-02

Euler's polyhedron formula

For a 2-dimensional polyhedron p that is a homological sphere,

$$V - E + F = 2$$

(number of **V**ertices, **E**dges, and **F**aces).

General statement

More generally, for an d-dimensional homological sphere p, we have

$$\sum_{j\geq 0}^{d} (-1)^j k_j = 1 - (-1)^d,$$

where k_j is the number of j-dimensional faces of p.

Combinatorial definition

A polyhedron is a finite sequence I of incidence relations with a boundary operator ∂_k for each dimension k is defined as

$$\partial_k(x) := \{y \in P_{k-1} | I_k(y, x)\}$$

Homology spheres

A polyhedron is a *homological sphere* if

- $\partial^2 = 0$
- Every vertex assumed to be incident with a virtual -1-dimensional 'face' (i.e., $\operatorname{im} \partial_0 = C_{-1}$, with C_{-1} being 1-dimensional)
- Every $(d-1)$-dimensional face is stipulated to be incident with a virtual d-dimensional 'whole polyhedron' (i.e., $\ker \partial_d = 0$)

Euler-Poincaré theorem

Theorem: In a chain complex $\langle C_k, \partial_k \rangle$, we have

$$\sum_k (-1)^k \, |C_k| = \sum_k (-1)^k \, |H_k|,$$

where $H_k = \text{im } \partial_{k+1} / \text{ker } \partial_k$.

First proof attempt

First proof attempt (with assistance from Claud
Code, Loogle, and Leanserch) took a couple of weeks.

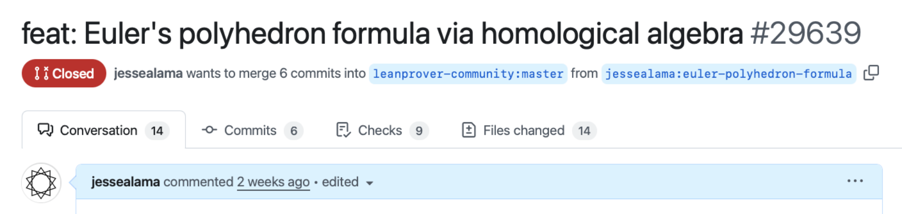
It made no attempt at linking up with Mathlib's chain complex machinery (every stated in terms of incidence relations and finite-dimensional vector spaces over Z_2),

It was also *ugly*, unpleasant, and gave me a headache.

2nd attempt

- I used the `ChainComplex` in Mathlib (fewer headaches, yay!).
- Used augmentation to 'properly' obtain the existence of the virtual -1-dimensional face
- I was feeling bold: I made a PR and posted about it on Zulip. Yay!

Not so fast...


feat: Euler's polyhedron formula via homological algebra #29639
Closed
jessealama wants to merge 6 commits into leanprover-community:master from jessealama:euler-polyhedron-formula
Conversation 14
Commits 6
Checks 9
Files changed 14
jessealama commented 2 weeks ago • edited

Reviewer feedback

- This work is combinatorial only; what about real polyhedra?
- This doesn't connect with larger discussions about formalizing such things, such as combinatorial maps, planar graphs, and the issue there of the Jordan curve theorem.
- *By the way:* Your formalization proves $\perp$!

Current status

- Euler-Poincaré for $\mathbb{Z}$-indexed chain complexes factored out into its own PR. (*In review*)

- Working on developing enough convex geometry to be dangerous (enough to associate a chain complex with a convex polyhedron and prove that they homology spheres).

Further work

- Exploring CW complexes as generalizations of ‘real’ polyhedra; goal is to get an Euler polyhedron formula in that setting.

- Combinatorial maps (should be straightforward to associate a combinatorial map with a convex polyhedron and a CW complex).