The Expressive Power of Description Logics with Numerical Constraints over Restricted Classes of Models

Franz Baader Filippo De Bortoli

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Description Logics: what are they?

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Description logics are logic-based knowledge representation languages.

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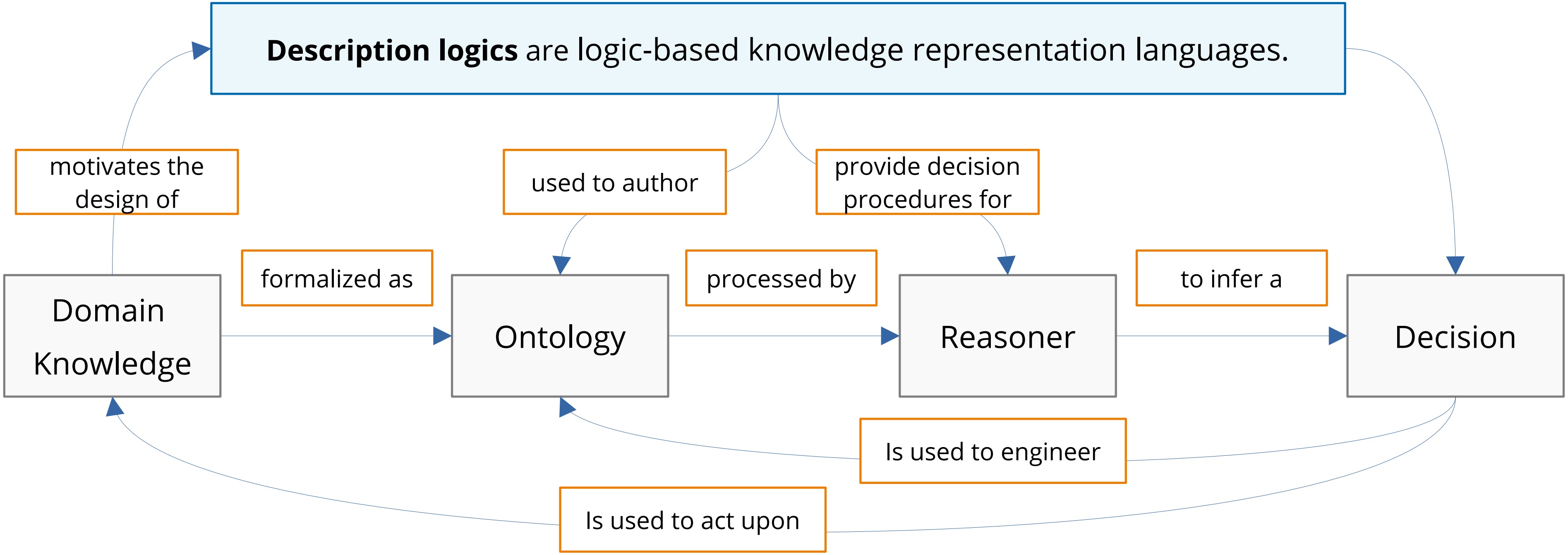
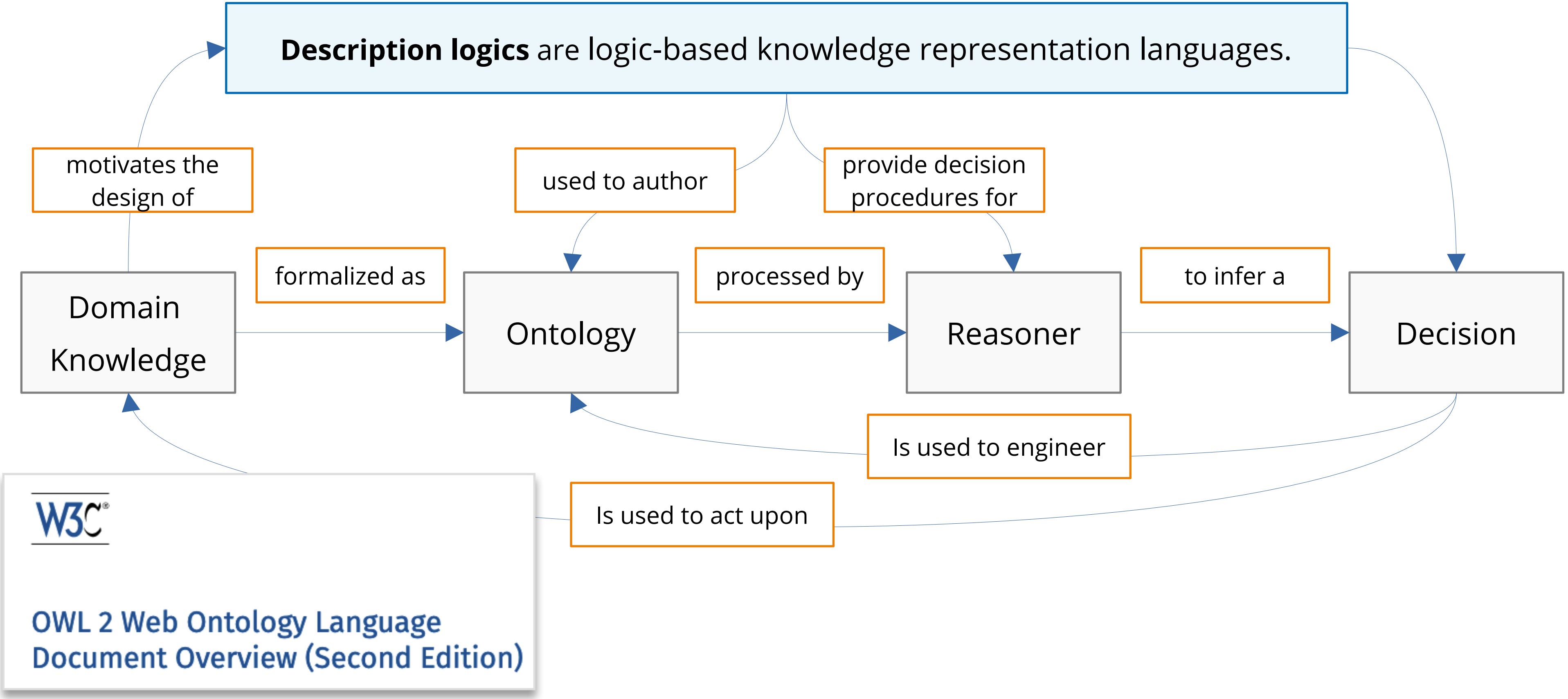


Description Logics: what are they?



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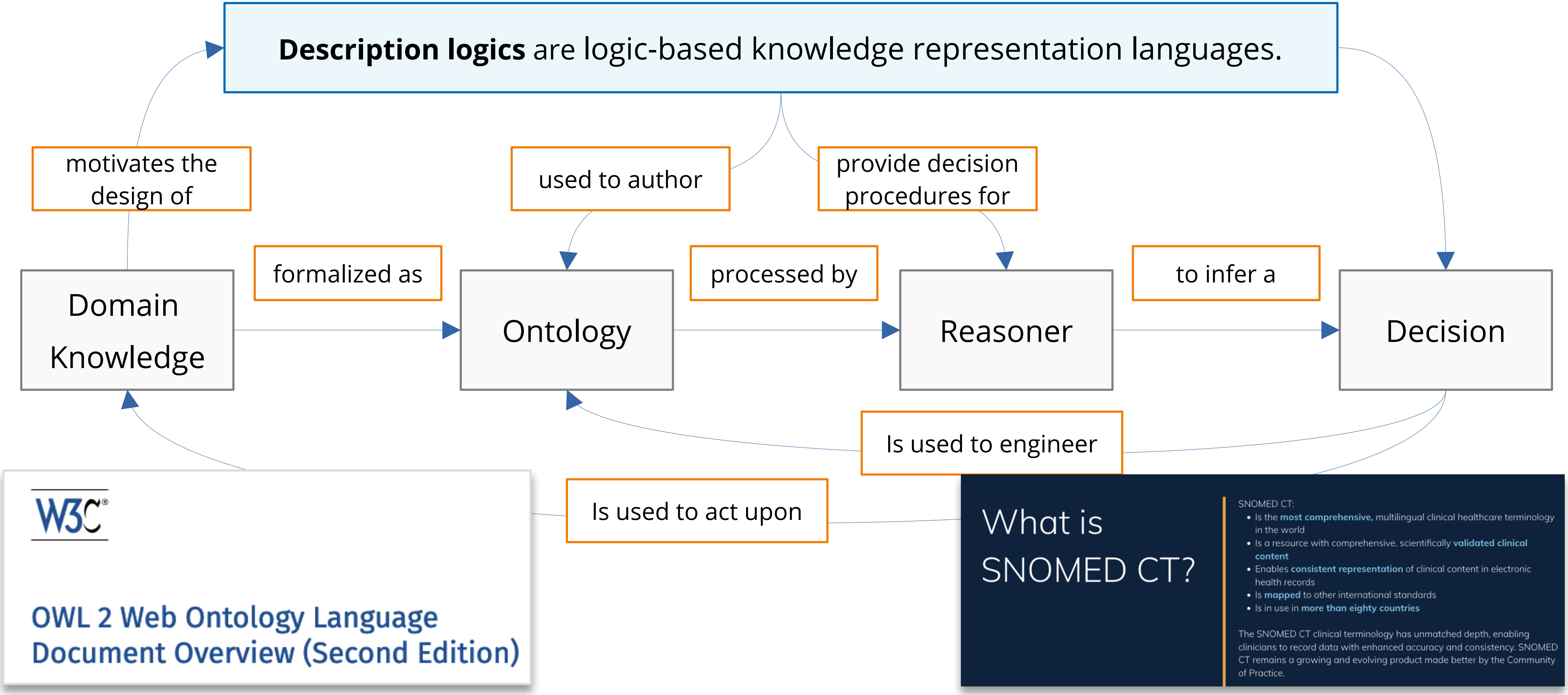

Basics of Description Logics

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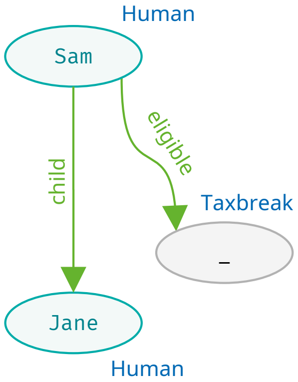

concept names (Human, TaxBreak), role names (child, eligible)

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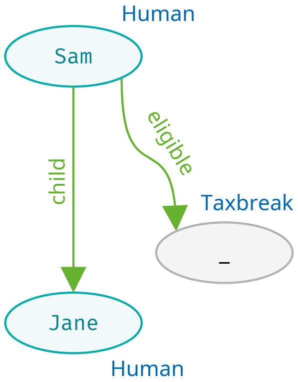


Concept descriptions

$C ::= A \mid \neg C \mid C \sqcap C \mid \exists r.C$ ← ALC

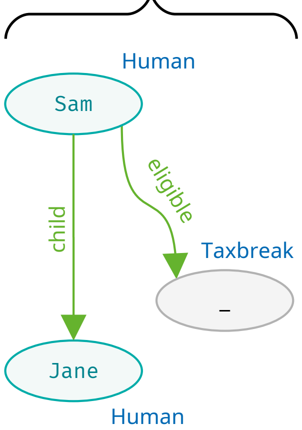
$$\overbrace{\text{Human} \sqcap \exists \text{child}.\text{Human}} \sqsubseteq \overbrace{\exists \text{eligible}.\text{TaxBreak}}$$

← **TBox** (*concept inclusions*)

concept names (Human, TaxBreak), role names (child, eligible)

Basics of Description Logics

interpretation



$\models$

Concept descriptions

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TBox (*concept inclusions*)

TBox consistency is ExpTime-complete for ALC ontologies.

concept names (Human, TaxBreak), role names (child, eligible)

Cardinality constraints in description logics

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Qualified number restriction

ALC + $(\geqslant n\ r.\ C)$ and $(\leqslant n\ r.\ C)$

ALCQ

$$\mathsf{Human} \sqcap \underbrace{(\geqslant 3\ \mathsf{child}.\mathsf{Human})}_{\text{Qualified number restriction}} \sqsubseteq \exists \mathsf{eligible}.\mathsf{TaxBreak}$$

Cardinality constraints in description logics

$\mathrm{ALC} + (\geqslant n\ r.\ C) \text{ and } (\leqslant n\ r.\ C) \longleftarrow \boxed{\mathrm{ALCQ}}$

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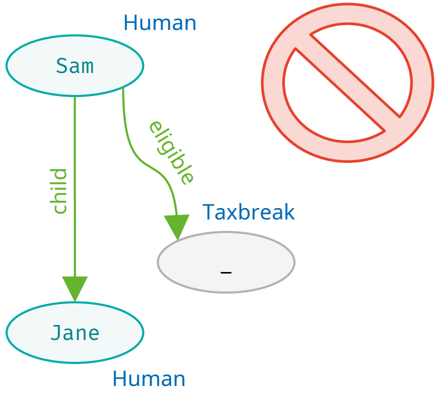

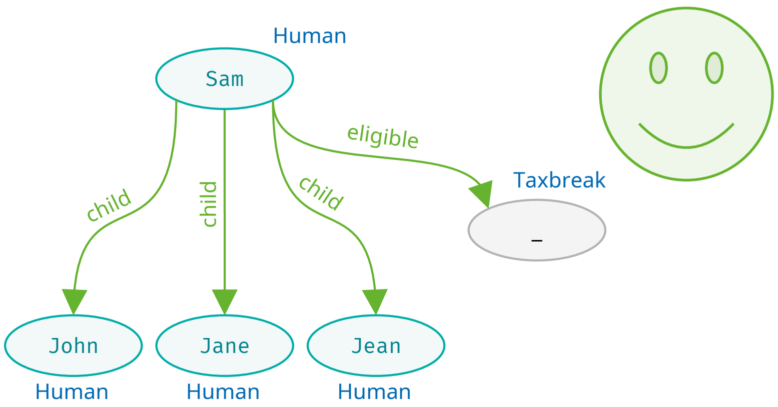

Cardinality constraints in description logics

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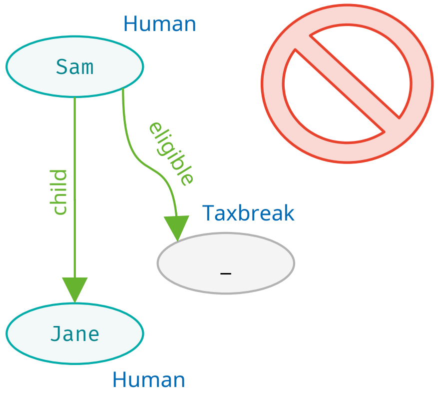

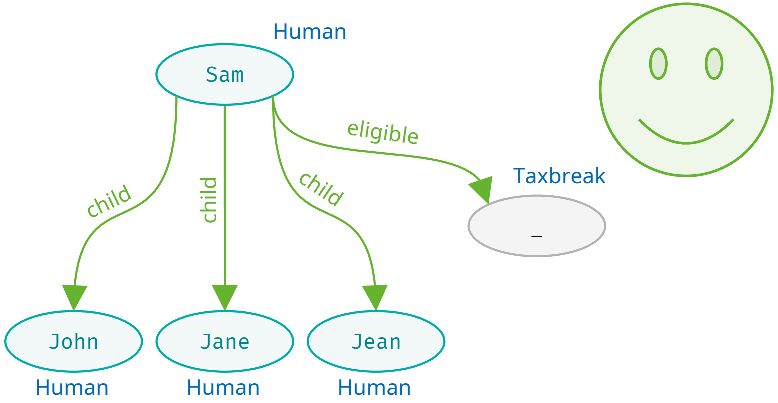

TBox consistency is ExpTime-complete in $\mathcal{ALCQ}$ (Tobies '00,'01) for unary and binary coding of numbers

Concrete domains

Concrete domains

$\text{Human} \sqcap (\geqslant 3\ \text{child}.\text{Human}) \sqcap \exists \text{salary}.{<_{100{,}000}} \sqsubseteq \exists \text{eligible}.\text{TaxBreak}$

$\text{Mother} \sqsubseteq \exists \text{age}, \text{child age}.{>}$

Feature names

Concrete domains

Concrete domain restrictions

ALC + concrete domain restrictions ← $\mathcal{ALC}(\mathfrak{D})$

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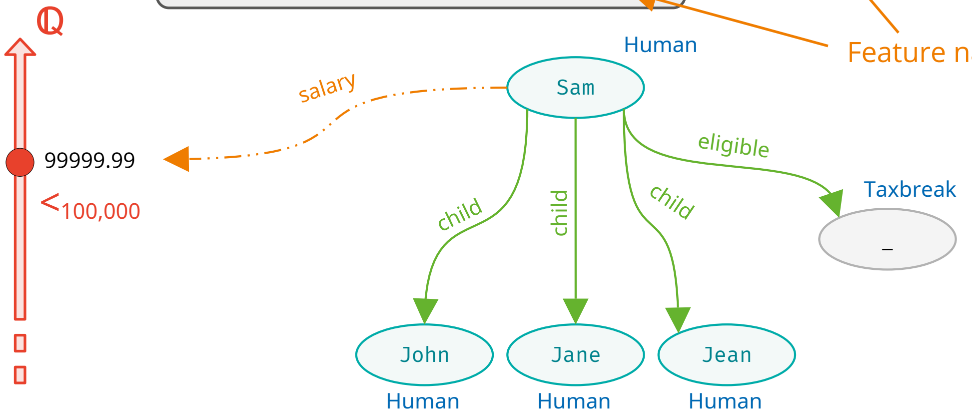

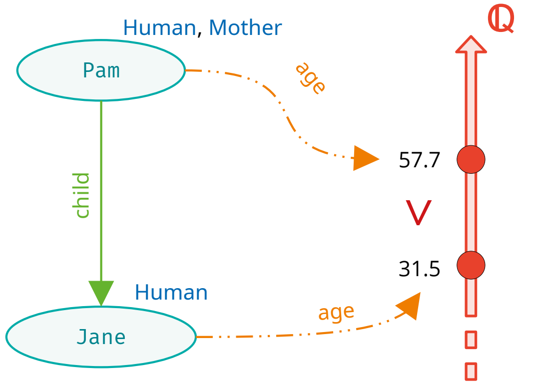

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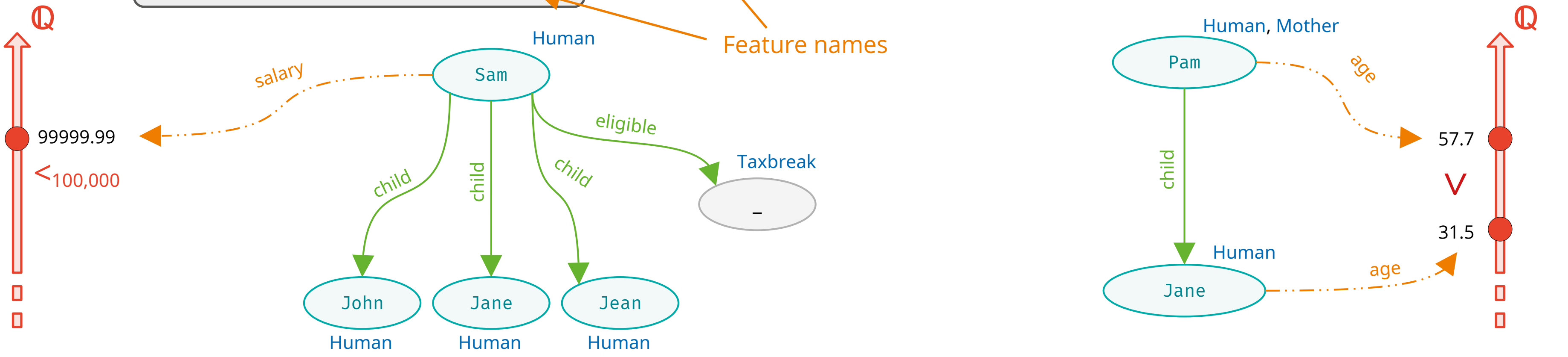

TBox consistency ExpTime-complete for ALC extended by integers with comparison (Labai, Ortiz & Šimkus '20) or rationals with comparisons (Borgwardt, D., Koopmann '24)

Expressive power of concept languages

$$\exists \text{child.Human} \equiv (\geqslant 1\ \text{child.Human}) \equiv \varphi(x) := \exists y.\ (\text{child}(x,y) \land \text{Human}(y))$$

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DL’
DL

“DL and DL’ are equivalent”

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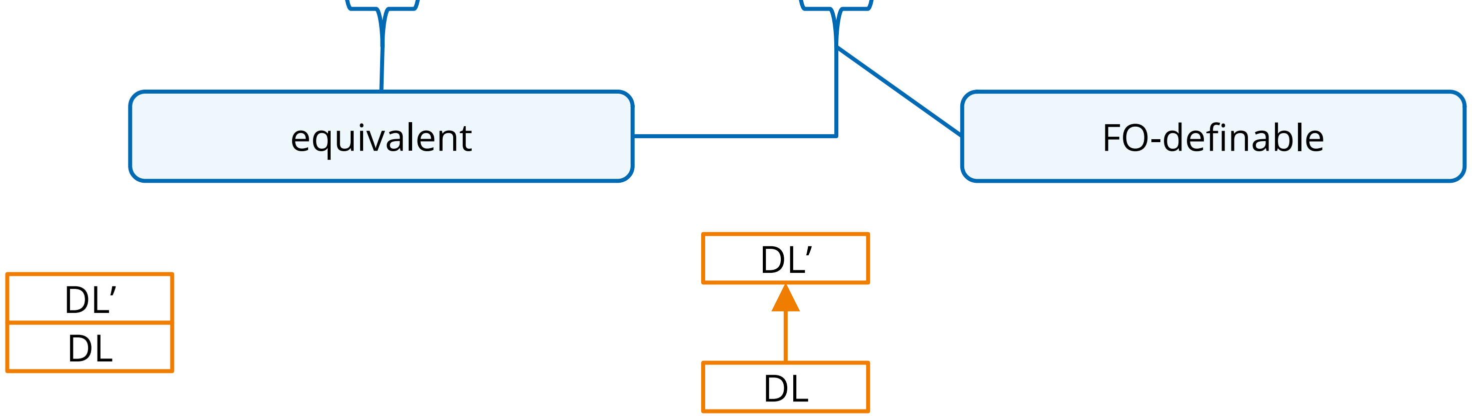

"DL and DL' are equivalent" "DL' strictly more expressive than DL"

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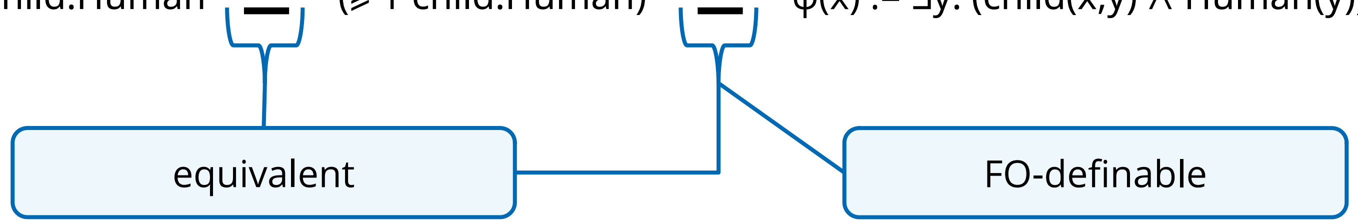

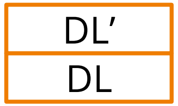

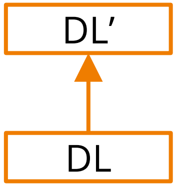

“DL and DL’ are equivalent” “DL’ strictly more expressive than DL” “DL is a first-order fragment”

Expressive power of concept languages

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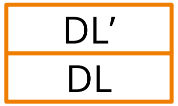

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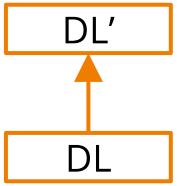

“DL’ strictly more expressive than DL”

“DL is a first-order fragment”

We assume that DL, DL’, FOL use *the same sets* of names.

Bisimulations, characterization and non-expressivity

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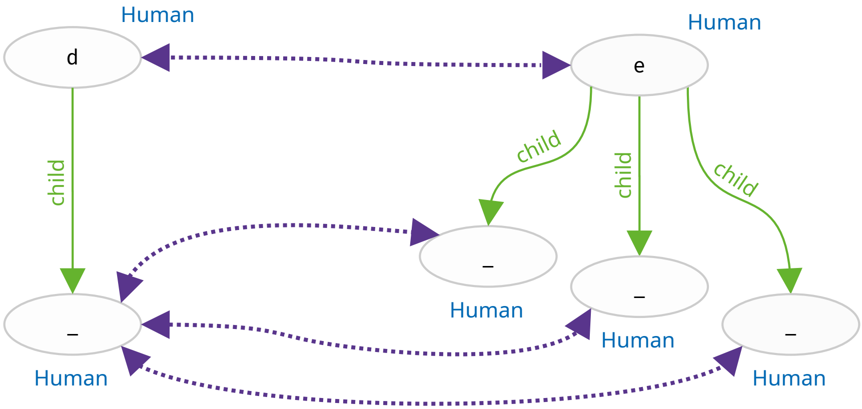
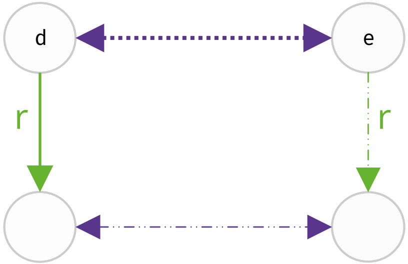


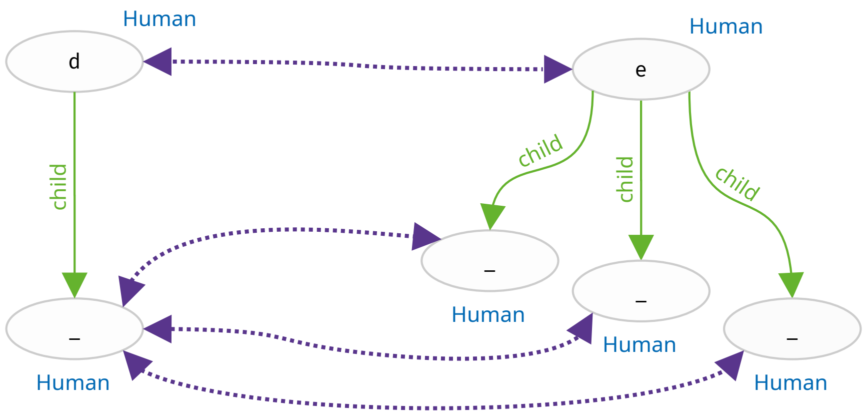
Bisimulation

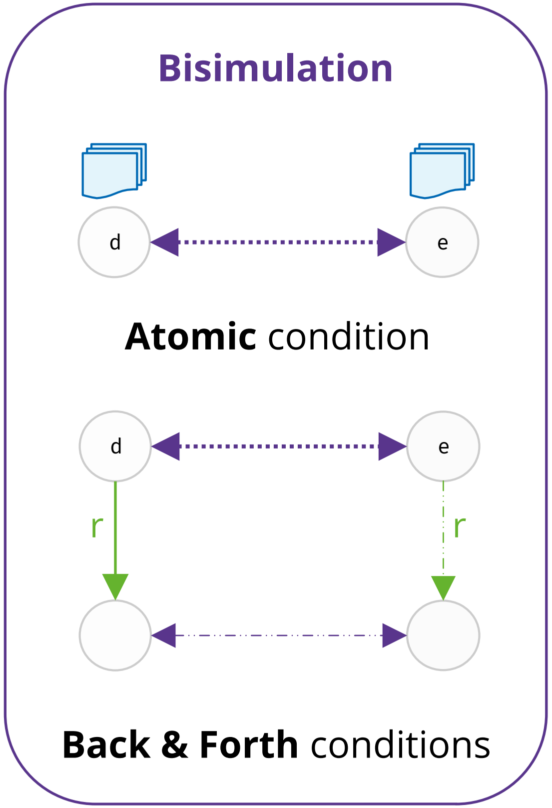
Atomic condition



Back & Forth conditions

Bisimulations, characterization and non-expressivity

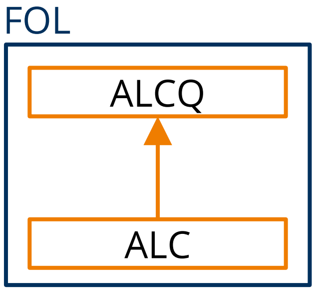


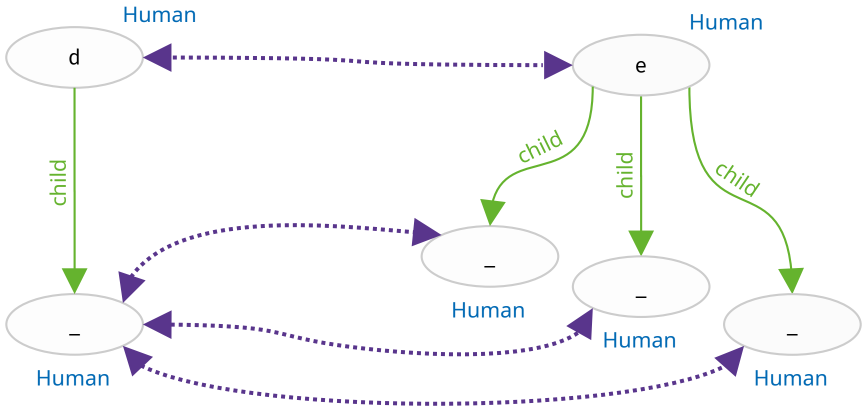


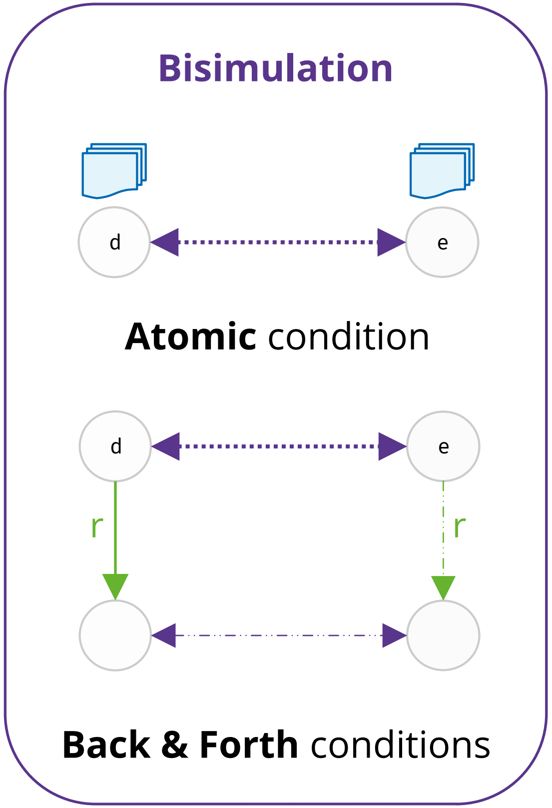
φ(x) is invariant under bisimulation ↔ φ(x) is equivalent to ALC concept

- Van Benthem '76: proof w.r.t. all interpretations
- Rosen '97: proof also w.r.t. finitely branching or finite interpretations

Bisimulations, characterization and non-expressivity







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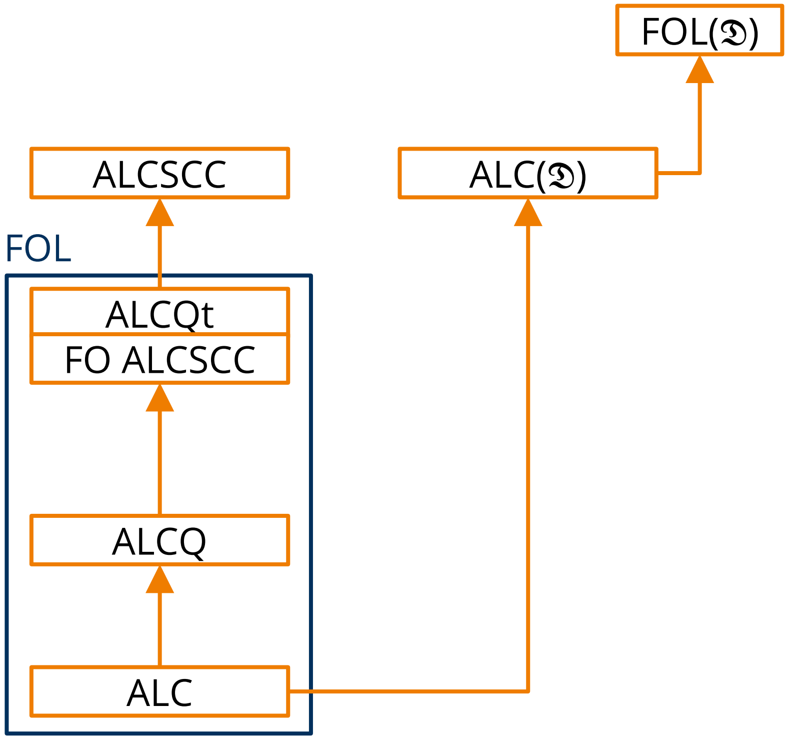
(⩽ 1 child. Human) is *not* invariant under bisimulation!

Outline

1) *Cardinality constraints*
2) *Concrete domains*



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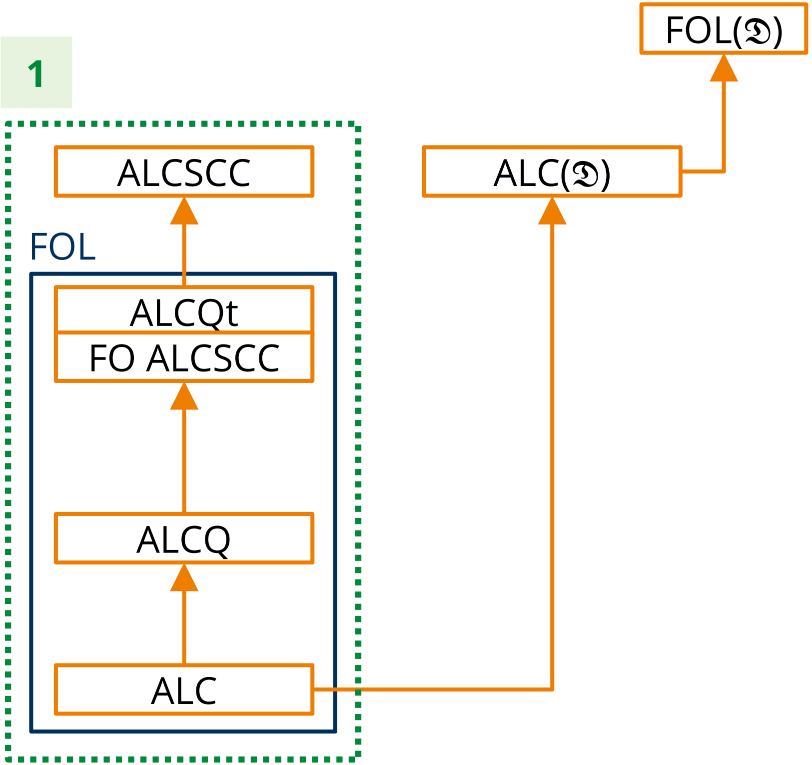

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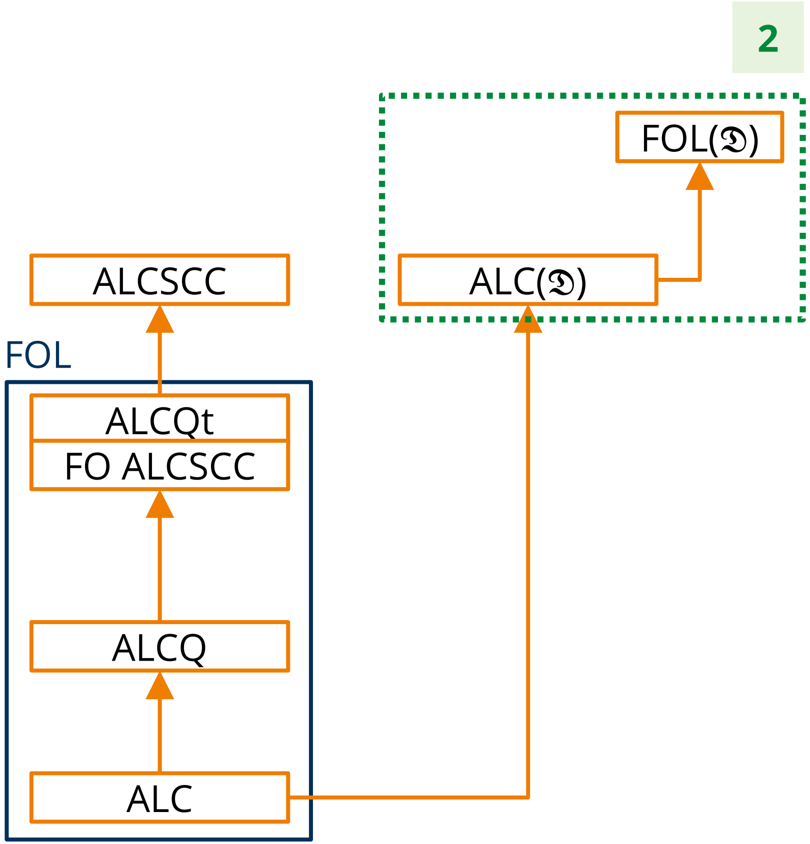
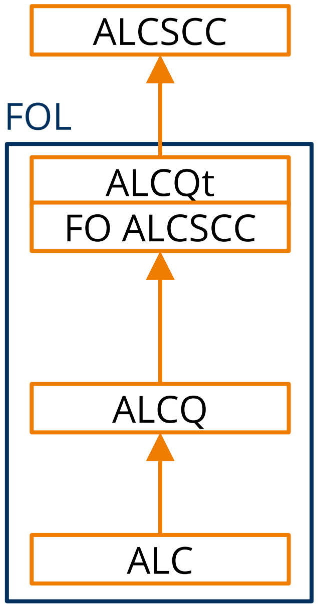


Cardinality constraints



The description logic ALCSCC

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ALC + successor restrictions based on QFBAPA

ALCSCC

The description logic ALCSCC

ALC + successor restrictions based on QFBAPA ← ALCSCC

$$\text{Human} \sqcap \underbrace{\text{succ}(|\text{pet} \cap \text{Dog}| = |\text{child} \cap \text{Human}|)}_{\textit{successor restriction}}$$

Successor restrictions evaluated w.r.t. all role successors of an individual

The description logic ALCSCC

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- QFBAPA: set and cardinality constraints over finite sets
- Satisfiability for QFBAPA is NP-complete
- **Baader & D. '19:** QFBAPA$^{\infty}$, like QFBAPA but with infinite sets, we show that it is NP-complete

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Baader '17: TBox consistency is ExpTime complete; defined only over finitely branching interpretations!

Baader & D. '19

- ALCSCC$^\infty$:= ALCSCC defined over all interpretations
- TBox consistency remains ExpTime-complete

Separation using *counting* bisimulation → ALCSCC$^\infty$

ALCQ → ALCSCC$^\infty$

Presburger bisimulation

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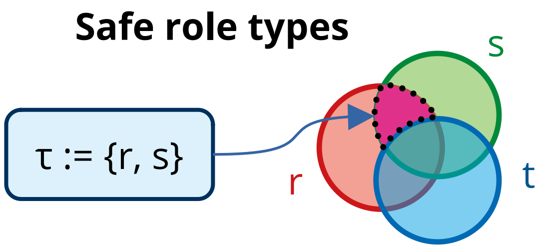
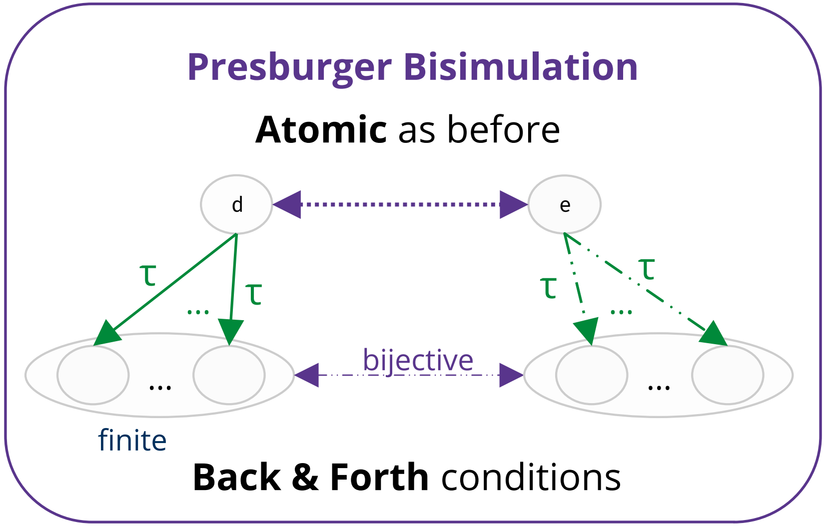
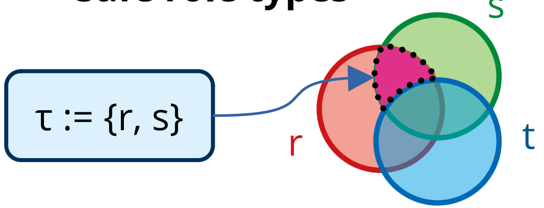




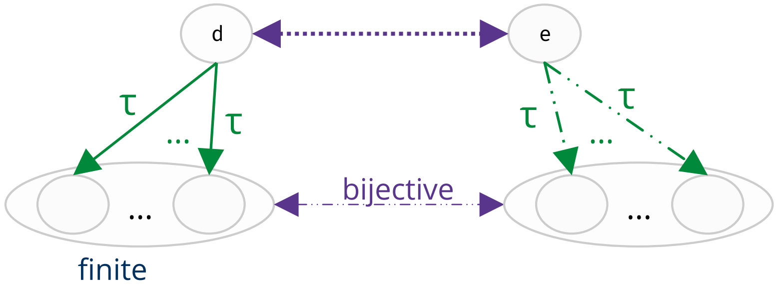
Presburger bisimulation

Safe role types



Presburger Bisimulation

Atomic as before



Back & Forth conditions

For all FOL formulae φ(x), the following are equivalent:

1) φ(x) is equivalent to some ALCSCC concept.
2) φ(x) is invariant under Presburger bisimulation.
3) φ(x) is equivalent to some ALCQt concept.

- **Baader & D. '19:** showed for all interpretations
- **FroCoS '25:** extended to finitely branching/finite

Presburger bisimulation

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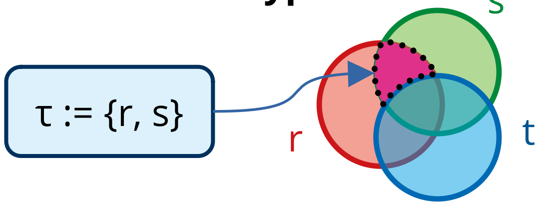

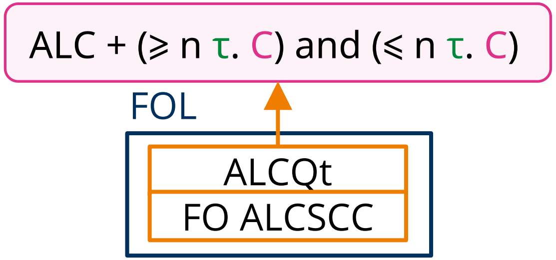

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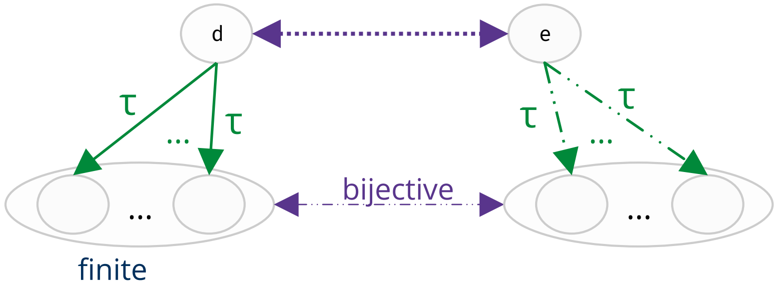

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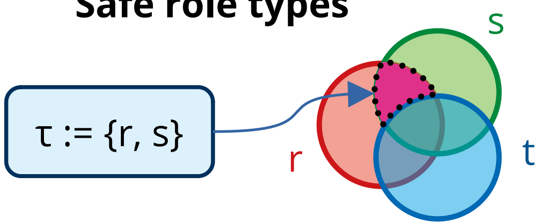
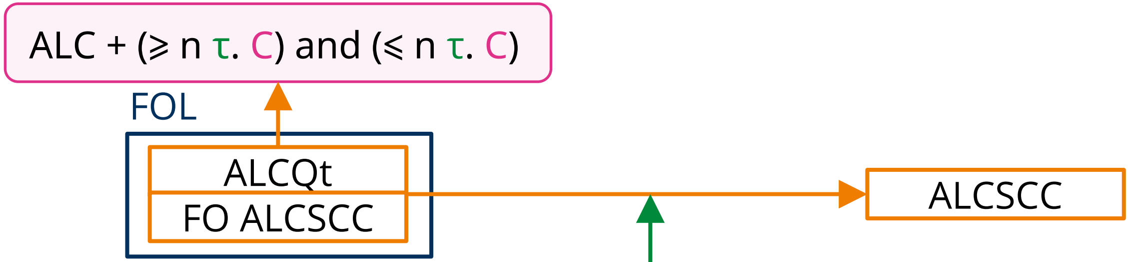
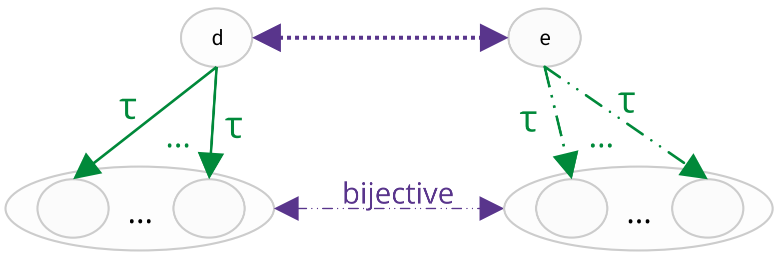




succ(|r ∩ A| = |r ∩ ¬A|) is not first-order definable!

Presburger Bisimulation

Atomic as before



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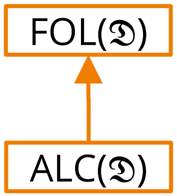
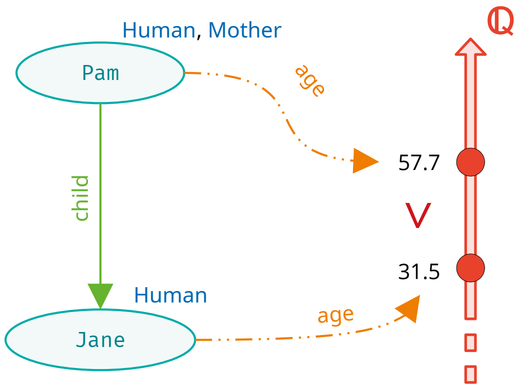
Concrete Domains



First-order logics with concrete domains (SAC '24)



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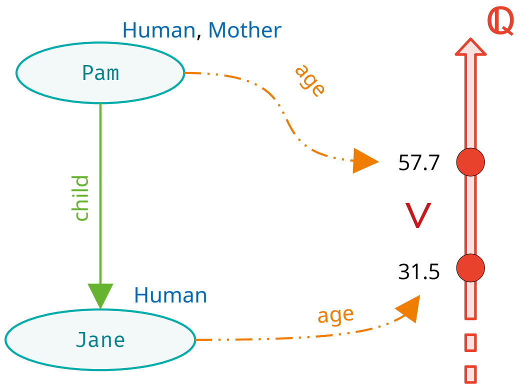
$\text{Mother} \sqsubseteq \exists \text{age}, \text{child age}.<$

ALC + concrete domain restrictions ← $\mathcal{ALC}(\mathfrak{D})$



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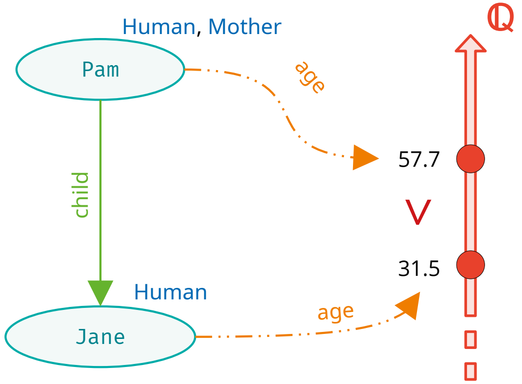
FOL + definedness predicates + concrete domain predicates ← FOL($\mathfrak{D}$)

ALC + concrete domain restrictions ← ALC($\mathfrak{D}$)



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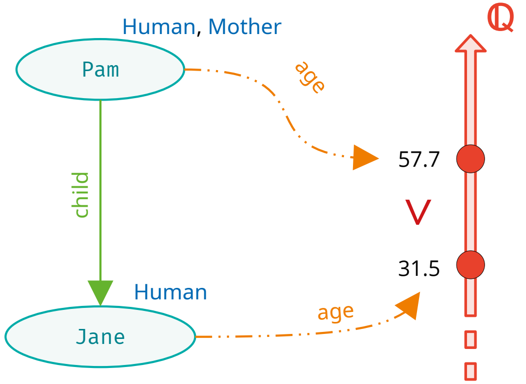
ALC + concrete domain restrictions ← ALC($\mathfrak{D}$)

$\forall x,y.\ \mathrm{Def}(id)(x) \wedge (x \neq y \rightarrow <(id,id)(x,y) \vee <(id,id)(y,x))$

"id acts as an injective function" (satisfiable)



$\exists x,y,z.\ (<(age,age)(x,y) \wedge <(age,age)(y,z) \wedge <(age,age)(z,x))$

"there is a <-cicle of age-values" (unsatisfiable)

First-order logics with concrete domains (SAC '24)

$\text{Mother} \sqsubseteq \exists \text{age}, \text{child age}.<$

FOL + definedness predicates + concrete domain predicates ← FOL($\mathfrak{D}$)

ALC + concrete domain restrictions ← ALC($\mathfrak{D}$)

Definedness predicate

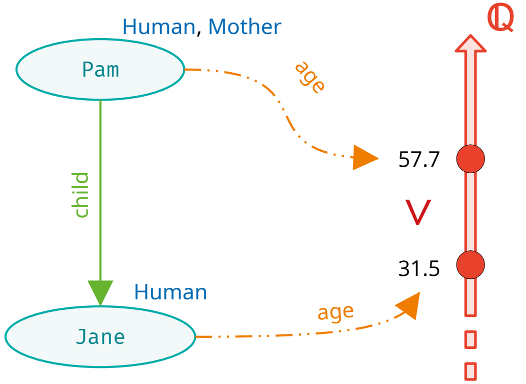
$\forall x,y.\ \text{Def}(\text{id})(x) \land (x \neq y \rightarrow <(\text{id},\text{id})(x,y) \lor <(\text{id},\text{id})(y,x))$

"id acts as an injective function" (satisfiable)

Concrete domain predicates

$\exists x,y,z.\ (<(\text{age},\text{age})(x,y) \land <(\text{age},\text{age})(y,z) \land <(\text{age},\text{age})(z,x))$

"there is a <-cicle of age-values" (unsatisfiable)



Concrete domains and negation

Concrete domains and negation

Conditions on $\mathfrak{D}$ that enable the expression of negated predicates in ALC($\mathfrak{D}$) or FOL($\mathfrak{D}$):

- **Weakly Closed Under Negation (WCUN):** complement of a k-ary predicate is a union of k-ary predicates

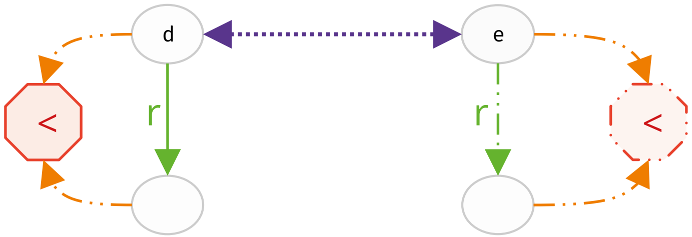
$$\neg(x = y) \text{ iff } (x < y) \vee (y < x)$$

Bisimulations for concrete domains

Bisimulations for concrete domains

$\mathfrak{D}$ **Bisimulation**

Atomic as before
Back & Forth as before



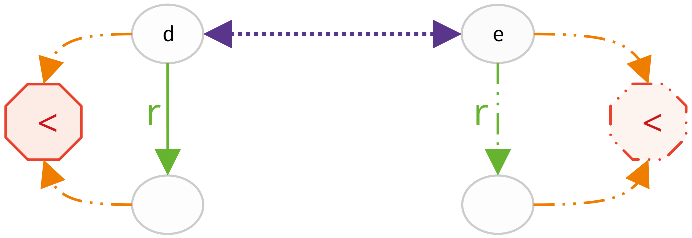
Feature conditions

if $(d,e) \in \rho$, then there is $(v_1,\dots,v_k) \in P^D$ with $v_1 \in p_1^{\mathcal{I}}(d), \dots, v_k \in p_k^{\mathcal{I}}(d)$ iff there is $(w_1,\dots,w_k) \in P^D$ with $w_1 \in p_1^{\mathcal{I}}(e), \dots, w_k \in p_k^{\mathcal{I}}(e)$.

Bisimulations for concrete domains

$\mathfrak{D}$ Bisimulation

Atomic as before
Back & Forth as before



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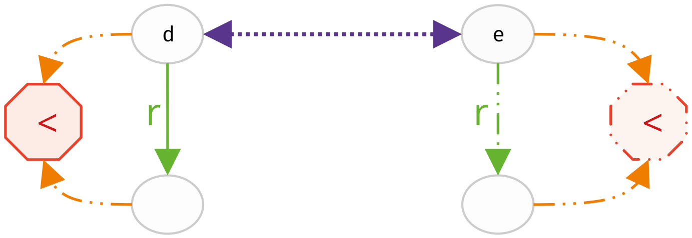
Bisimulation and non-expressivity:

- extensions of ALC with different concrete domains
  - e.g. ALC($\mathbb{Q}$, $+_1$) and ALC($\mathbb{Q}$, $+_2$) "orthogonal"
- different extensions of ALC w/ same concrete domain
  - e.g. ALC($\mathbb{Q}$, <) cannot express restriction with constraint systems e.g. $\exists r\ f, r\ f, r\ f.(x < y \land y < z)$

Bisimulations for concrete domains

$\mathfrak{D}$ Bisimulation

Atomic as before
Back & Forth as before



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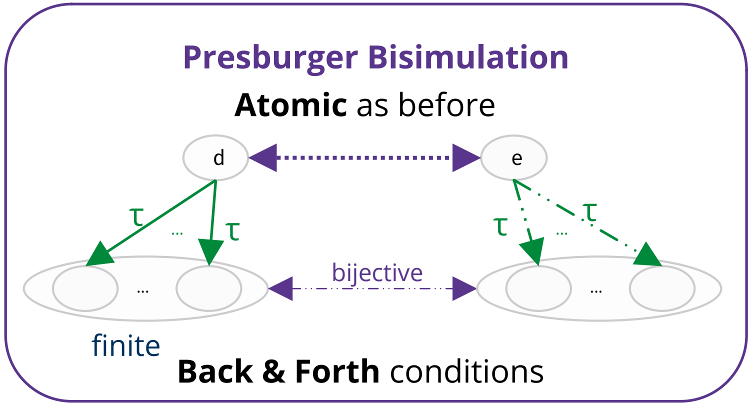
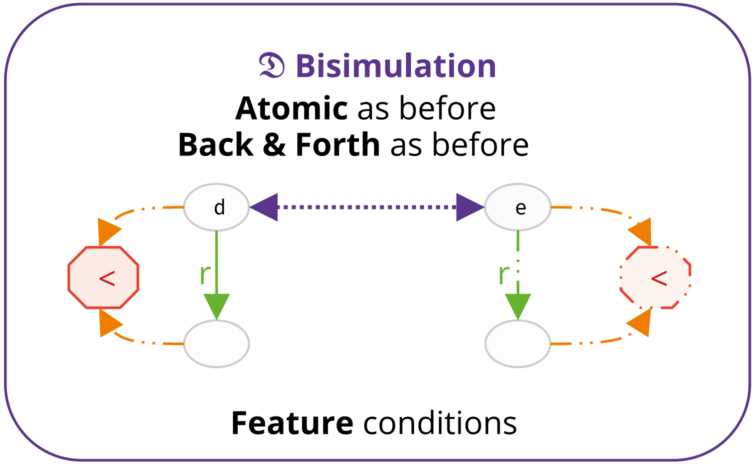
Assume finite sets of names and let be WCUN and have finitely many relations. For all FOL($\mathfrak{D}$) formulae $\varphi(x)$, the following are equivalent:

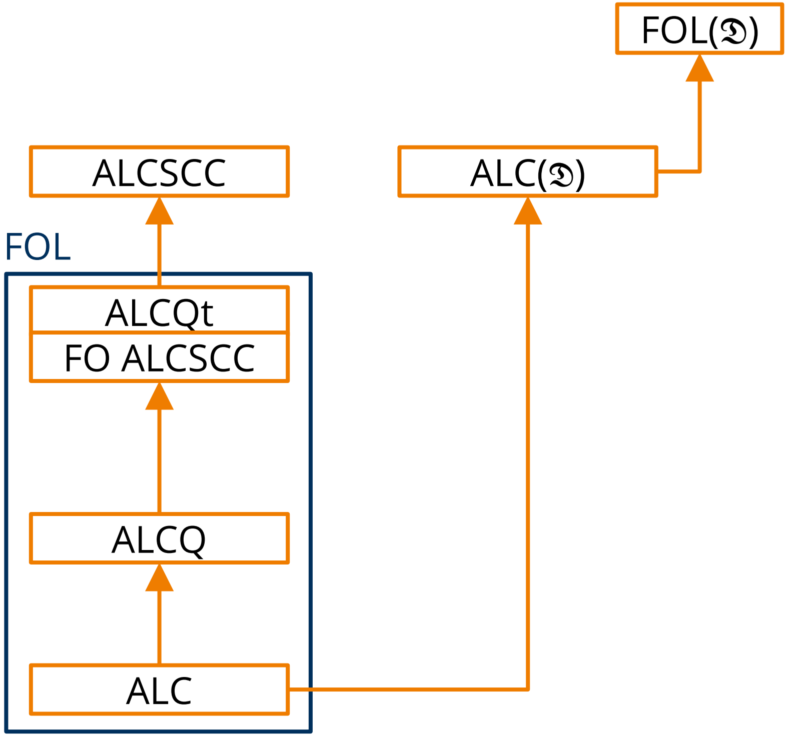
1) $\varphi(x)$ is invariant under bisimulation.
2) $\varphi(x)$ is equivalent to some ALC($\mathfrak{D}$) concept.

- **FroCoS '25:** showed w.r.t all interpretations as well as finitely branching/finite ones

Summary: expressive power







From here on...

- Expressive power of ontologies with numerical constraints*
  - e.g. $ALCSCC^{\infty}$ TBoxes using global Presburger bisimulation (Baader, D. ‘20)
- Expresive power when combining cardinality constraints and concrete domains (CADE ‘25)
- Different notion of expressive power, e.g. conservative extensions

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Thanks! :-)

ALCSCC($\mathfrak{D}$) and feature roles (CADE '25)

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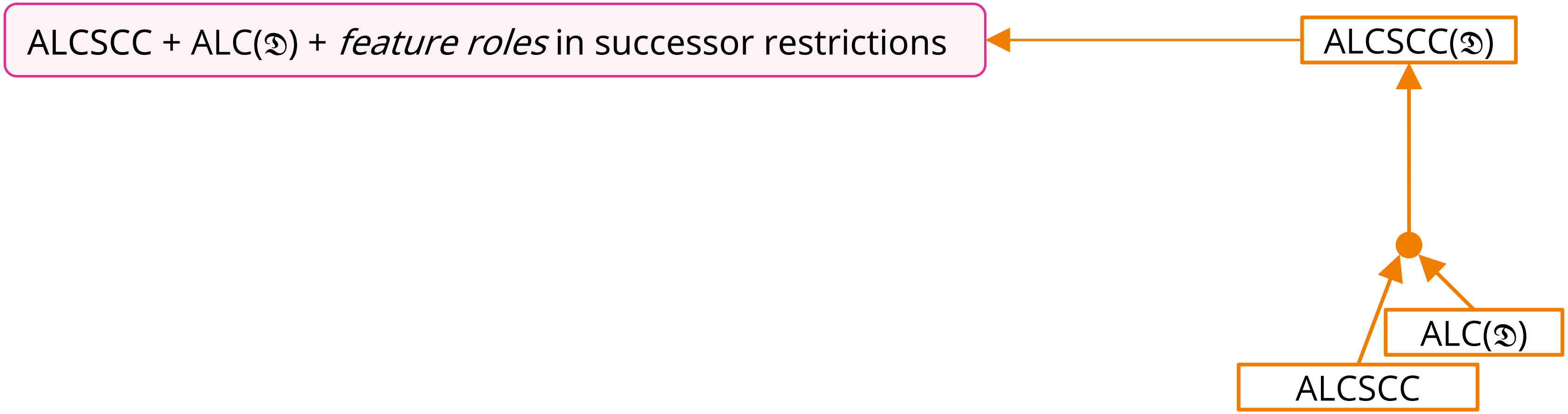
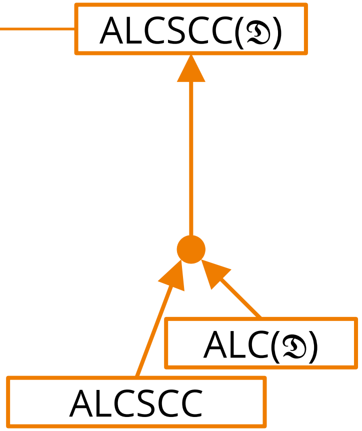


ALCSCC($\mathfrak{D}$) and feature roles (CADE '25)

ALCSCC + ALC($\mathfrak{D}$) + *feature roles* in successor restrictions



$\mathrm{succ}(|\mathrm{child} \cap (\mathrm{salary} < \mathtt{next}\ \mathrm{salary})| > |\mathrm{child} \cap (\mathrm{salary} < \mathtt{next}\ \mathrm{salary})^c|)$

"the salary of this individual is smaller than that of the majority of its children"

ALCSCC($\mathfrak{D}$) and feature roles (CADE '25)

ALCSCC + ALC($\mathfrak{D}$) + *feature roles* in successor restrictions

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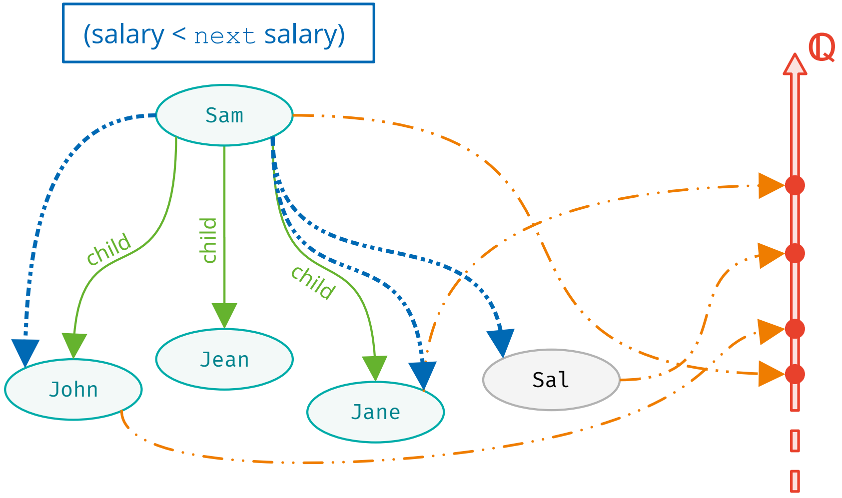

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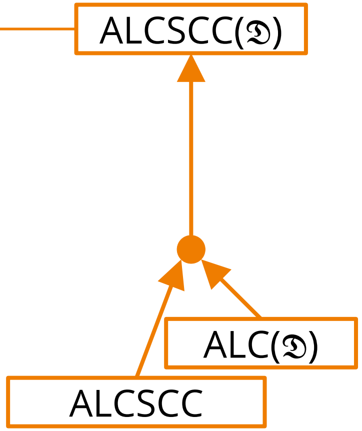

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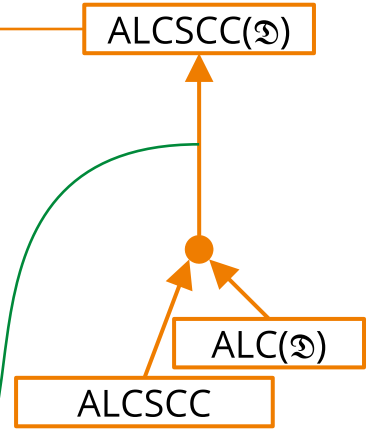
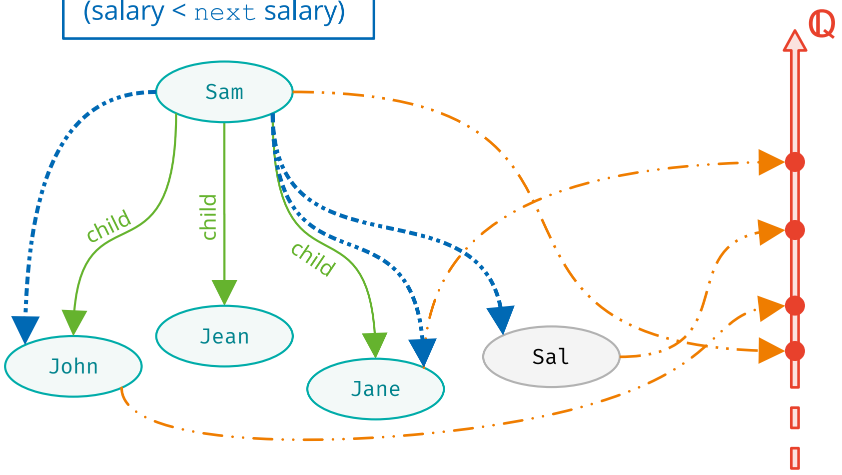
$\mathrm{succ}(|\mathrm{child} \cap \underbrace{(\mathrm{salary} < \mathtt{next}\ \mathrm{salary})}_{\text{feature role}}| > |\mathrm{child} \cap \underbrace{(\mathrm{salary} < \mathtt{next}\ \mathrm{salary})}_{\text{feature role}}{}^{c}|)$



(salary < $\mathtt{next}$ salary)

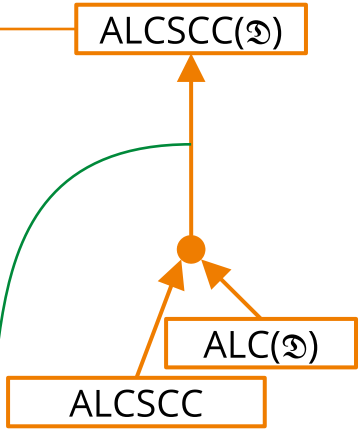


"the salary of this individual is smaller than that of the majority of its children"

It cannot be expressed without feature roles!

ALCSCC($\mathfrak{D}$) and feature roles (CADE '25)

ALCSCC + ALC($\mathfrak{D}$) + *feature roles* in successor restrictions



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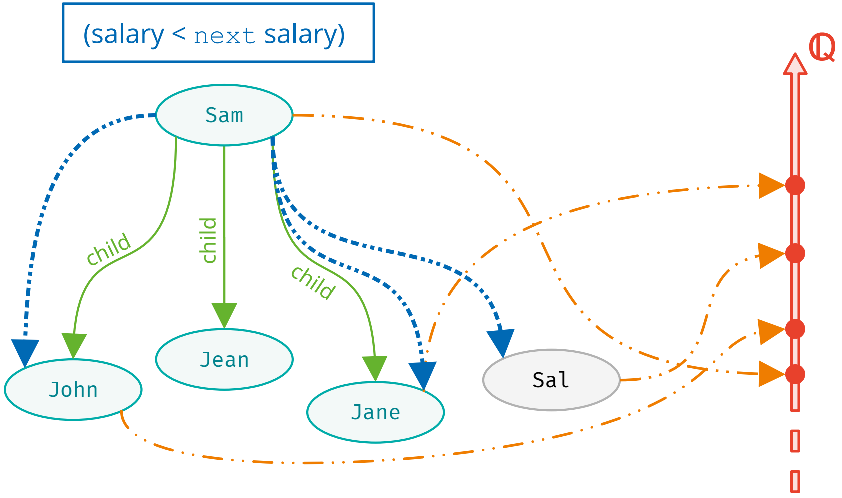

"the salary of this individual is smaller than that of the majority of its children"

It cannot be expressed without feature roles!

$$(\exists \mathrm{salary}, \mathrm{child}\ \mathrm{salary}.<) \sqsubseteq (\mathrm{succ}(|\mathrm{child}| \leq 0))$$

Inconsistent, due to the interaction of cardinality constraints and concrete domain!