An Analytic Representation of the Semantics of First-Order S5

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Roadmap

Why Interpolation Matters

Craig interpolation: if $A \models B$, then $\exists I$ s.t.

$$A \models I \quad \text{and} \quad I \models B,$$

with I using only the common vocabulary.

- **Intuition**: The interpolant acts as a bridge between premises and conclusion.
- **Restricting inference**: Interpolation guarantees that only the relevant parts are affected by contradiction.
- **Philosophically**, it resonates with the principle of relevance: arguments should not rely on symbols or concepts alien to both premises and conclusion. Only what is *shared* with the goal matters for deriving the goal.

Applications: Keeping Inconsistencies Local

- database reasoning: local errors stay local; queries only see shared vocabulary.
- In verification/modular reasoning: components communicate via their interfaces (the shared symbols).
- Resolution with Set of Support (SoS): In resolution theorem proving, interpolation is connected with set of support strategies, which restrict proofs to relevant clauses and avoid unnecessary explosion of contradictions.

Algebraic View: connect syntactic logic with algebraic structure

- In propositional logic, interpolants often correspond to natural algebraic constructions (e.g., lattice-theoretic factorization (joins and meets) through the shared subalgebra).
- In propositional logic, interpolation properties can often be determined and classified using the foundational work of **Maksimova**.
- algebrization of the first-order logic fails as it is impossible to determine what first-order logics interpolate.

Beth Definability: Implicit $\Rightarrow$ Explicit

Beth Definability (Classical FOL)

If a predicate P is *implicitly* definable from a theory T (its extension is fixed in all models of T), then there exists a formula $\varphi(\vec{x})$ (in the language of T without P) such that $P(\vec{x}) \leftrightarrow \varphi(\vec{x})$ is entailed by T.

- Interpolation is closely tied to Beth Definability.
- **Beth's Theorem (FOL):** implicit (*semantic*) $\iff$ explicit(*syntactic*).
- **Outside FOL:** Beth may fail (e.g. modal logics, quantified S5).

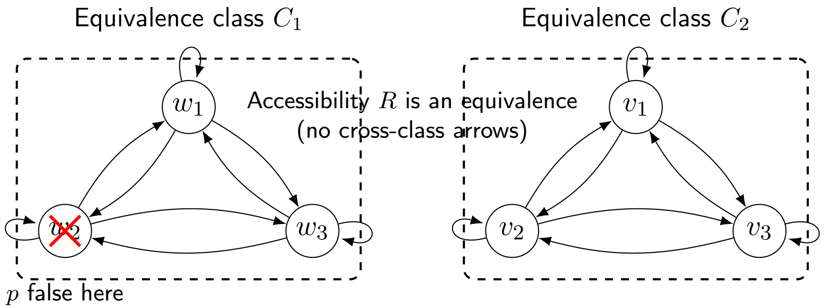
Motivation

1. interpolation holds in classical FOL and propositional logic, but fails in many other logics (modal, intuitionistic, arithmetic). This motivates research.
2. establishing interpolation in first-order logics is significantly more challenging, even in systems where the propositional case is well understood.
3. A particularly intriguing example of the failure of interpolation (and of Beth's definability theorem) arises in quantified S5.
    1. $p \supset \Diamond \forall x(F(x) \supset \Box(p \supset \neg F(x)))$,
    2. $\neg p \supset \Box \exists (F(x) \wedge \Box(\neg p \supset F(x)))$.
4. It is shown by **Kit Fine** that p is implicitly definable in T, but not explicitly definable.
5. propositional S5: Craig Interpolation + decidable (easy to show).

Quantified S5 basics

- $\mathcal{QML}$ language of quantified S5; No functions or individual constants.
- Kripke models $M = (W, R, D, V)$ with R an equivalence relation (S5) and **constant domain** D.
- All worlds in an equivalence class are mutually accessible, a formula true in one world is true in all worlds of the class. Read $\Box$ and $\Diamond$ as $\forall/\exists$ quantification over worlds.
- **rigid variables**: each variable designates the same object across all possible worlds.
- However, while terms refer rigidly, predicate interpretations may vary from world to world, their extensions are world-dependent.
- S5 validities: $\Box\Box A \leftrightarrow \Box A$, $\Diamond\Box A \leftrightarrow \Box A$.

S5 as Equivalence Classes (Kripke Semantics)



$\Box p$ is **false** at any world in C_1 because p is false at w_2.

A countermodel needs only **one** equivalence class: if p is false at some world of the class, then $\Box p$ fails everywhere in that class.

In S5, each equivalence class behaves like a "cluster" of mutually accessible worlds. Countermodels live inside a single cluster.

S5: Interpolation & Definability

- Quantified S5
  - Neither interpolation nor Beth definability hold in general.
  - Why?
    - No analytic sequent calculus with cut elimination.
    - Proof-theoretic obstacle: no syntactic route to interpolants.
- Only fragments recover good properties

General Interpolation Strategy

- embed S5 into a richer **two-sorted** first-order logic $\mathcal{SL}$
  - **World sort:** variables for worlds (restricted).
  - **Object sort:** quantifiers over individuals.
- **Separated terms:** no mixing of sorts, no world-function symbols; single world variable in the fragment.
- use proof theory there: **two-sorted** sequent calculus with cut elimination, a **mid-sequent theorem** and a **Maehara-style lemma** yield *interpolants* in the two-sorted logic.

what formulas correspond to the modal formulas in $\mathcal{SL}$

$$\begin{tikzcd} \text{World sort } W \arrow[d, "P'(w, \boldsymbol{x})"] \\ \text{Object sort } D \end{tikzcd}$$

Sorts

- World variables: $u, v, w, \ldots$
- Object variables: $x, y, z, \ldots$

Predicates

- Each modal n-ary predicate P becomes $P'(w, x_1, \ldots, x_n)$ (world first).
- **Formulas**: with arbitrary object quantifiers but with one variable class of world quantfiers.

Models

- $\mathcal{SL}$-model (W, D, V) with $V(P') \subseteq W \times D^n$.
- Validity is preserved by the translation (under the obvious correspondence of models).

Standard Translation ST_v

Fix a distinguished world variable v.

$$
\begin{aligned}
\mathrm{ST}_v(P(\bar{x})) &:= P'(v, \bar{x}) \\
\mathrm{ST}_v(\neg\varphi) &:= \neg\mathrm{ST}_v(\varphi) \qquad \text{(and similarly for } \wedge, \vee, \rightarrow\text{)} \\
\mathrm{ST}_v(\forall x\, \varphi) &:= \forall x\, \mathrm{ST}_v(\varphi) \\
\mathrm{ST}_v(\Box\varphi) &:= \forall w\, \mathrm{ST}_w(\varphi) \quad (w \text{ fresh world}) \\
\mathrm{ST}_v(\Diamond\varphi) &:= \exists w\, \mathrm{ST}_w(\varphi)
\end{aligned}
$$

Key fact: As all worlds are connected to all worlds and we work with constant domains, it consequently does not matter from which world variable v we start the translation ST_v if every subformula is in the scope of some modal operator.

Example

Translate $\Box(\forall x\, A(x) \to \Diamond\exists y\, A(y))$.

$$\begin{aligned}\mathrm{ST}_v\big(\Box(\forall x\, A(x) \to \Diamond\exists y\, A(y))\big) &= \Box(\forall x A(x) \supset \exists w \exists y A'(w,y)) \\ &= \forall v(\forall x A'(v,x) \supset \exists w \exists y A'(w,y)).\end{aligned}$$

$\Box/\Diamond$ become first-order quantifiers over the world sort.

In translations of modal formulas, each subformula is always bound by the innermost world quantifier that it is in the scope of.
Thus, formulas in $\mathcal{SL}$ in which world-quantifiers cross-bind express relations between worlds which cannot be expressed in modal logic.

One-world-variable fragment $\mathcal{SL}^v$

- **Define**: $\mathcal{SL}^v$ contains the formulas in the two-sorted logic $\mathcal{SL}$ in which every subformula is only bound by the innermost world quantifier. The only free world variable that might appear in a subformula is v, in which case the subformula is not in the scope of any other world-quantifier.
- **lemma**: For every quantified modal S5-formula φ, its translation $ST_v(\varphi)$ belongs to $\mathcal{SL}^v$ that can be expressed with a single bound world variable. In contrast, every formula in $\mathcal{SL}^v$ is the translation $ST_v(\varphi)$ of some formula φ in $\mathcal{QML}$.
- **Characterization:** $Re_v : \mathcal{SL}^v \longrightarrow \mathcal{QML}$ for every formula φ in $\mathcal{SL}^v$, - $\varphi = ST_v(Re_v(\varphi))$ and thus it is the translation of the formula $Re_v(\varphi)$ in $\mathcal{QML}$.
- A modal formula φ is valid in (constant domain) S5 if and only if its translation φ' is valid in the two-sorted first-order logic.

Proof-theoretic Properties of $\mathcal{SL}$

- Two-sorted sequent calculus; rules are the usual first-order ones, respecting sorts.
- **Cut elimination** holds for $\mathcal{SL}$.

Extended Mid-Sequent Theorem (EMST)

For a cut-free $\mathcal{SL}$ proof of a sequent with prenex world quantifiers, there is a *mid-sequent* such that:

- above it: only world-quantifier rules and propositional rules;
- below it: only object-quantifier rules and structural rules.

Intuition

The two sorts (world vs. object) can be permuted so that modal reasoning is *stratified* from object-level quantification.

Proof Sketch

1. **Cut elimination** for the two-sorted calculus (standard modal structural steps; separation ensures admissibility).
2. **Permutation of rules**: show modal and object quantifier inferences permute; prove no cross-sort dependency.
3. **Prenexing within sorts**: use invertibility/admissibility to float object quantifiers to one side, world quantifiers to the other.
4. **Extract mid-sequent**: the split yields a middle line from which an interpolant (in the shared signature) can be read.

Maehara-Style Lemma (Two-Sorted Version)

Setup / Definitions

Given a cut-free derivation of $\Gamma \Rightarrow \Delta$, fix a partition $(\Gamma_1 \mid \Gamma_2 \Rightarrow \Delta_1 \mid \Delta_2)$. Construct I should satisfy:

1. $\Gamma_1 \vdash I$ and $I, \Gamma_2 \vdash \Delta$ (or symmetrically with Δ_2),
2. **Language condition:** non-logical symbols of I lie in the intersection of those on both sides, *sortwise*
3. **Sort-respecting:** I is well-sorted (no mixed terms; separation guarantees this).

What We Get

- By Maehara's lemma, **Craig interpolation** holds in $\mathcal{SL}$.
- **Craig Interpolation** for the separated, prenex object-quantifier fragment $\mathcal{F}^{\mathrm{pren}}_{\mathrm{obj}}$.
- By duality, Craig Interpolation for the prenex world-quantifier fragment $\mathcal{F}^{\mathrm{pren}}_{\mathrm{wld}}$.
- Caveat: interpolants obtained in $\mathcal{SL}$ may fall *outside* the image fragment $\mathcal{SL}_v$ $\rightarrow$ they may not retranslate to S5.

- Maehara gives an $\mathcal{SL}$ interpolant I for $A \rightarrow B$.
- I may require cross-binding of world quantifiers or other constructions outside $\mathcal{SL}_v$.
- Such I has no equivalent formula in standard S5 syntax (echoing Fine's phenomenon).

Example

$\Box A(c) \to \exists x\, \Box A(x)$ is valid, but $\exists x\, \Box A(x) \to \Box \exists x\, A(x)$ is not.

Sketch.

This is a standard failure of modal quantifier shifts in constant domain semantics. While

$$\Box A(c) \supset \exists x \Box A(x)$$

holds due to domain constancy,

$$\exists x \Box A(x) \supset \Box \exists x A(x)$$

does not, because the witness for the existential quantifier in the accessible world may not be fixed across all worlds. A countermodel can be given with two worlds where the existential is satisfied in one, but the boxed existential fails globally. $\Box$

Note that therefore Craig's interpolation theorem holds for all valid implications, however the interpolant is not always re-translatable in the language of quantified S5.

Interpolation Results

Proposition

Any interpolant obtained in $\mathcal{SL}$ for Kit Fine's counterexample is not equivalent to any formula in the syntax of quantified S5.

Proof.

From the cut-free proof in $\mathcal{SL}$, Maehara's lemma yields an interpolant I. But this formula contains quantifier structures or term dependencies (from the two-sorted framework) that cannot be translated back into any formula of quantified S5. Thus, while I exists in $\mathcal{SL}$, there is no I' in the syntax of S5 such that $I = I'$. □

General Strategy

- In the **two-sorted setting** (world sort + object sort), interpolation always holds (mid-sequent theorem + Maehara lemma).
- But: interpolants are guaranteed only in the two-sorted logic, *not necessarily* in the modal fragment.
- To obtain interpolation for the modal fragment:
  1. Translate modal formulas to the two-sorted logic.
  2. Derive an interpolant there.
  3. Retranslate the interpolant back into the modal language.
- Success depends on whether retranslation is possible.

Case 1: Propositional S5

- Fact: **Propositional S5 has interpolation**.
- Why show it? Serves as a simple example of the method.
- Translation: Each modal propositional formula φ becomes a **monadic first-order formula** over the world sort.
- Result: The interpolant is also **monadic**.

Translation of propositional modal formulas yields monadic formulas in the world sort. Confinement of innermost quantifiers produces monadic interpolants that translate back to propositional S5 interpolants.

Normal Form Trick

- Existential quantifiers may arise in the translation.
- Strategy: rewrite formulas so that *existentials are confined to atomic world predicates*.
- Achieved by transforming into a disjunction of conjunctions (DNF-like step).
- Existentials then only bind atomic predicates, keeping the interpolant **monadic**.

Case 2: Weak-modal fragments

Two-sorted interpolant without adding quantifiers; normalize to a single free w; close modally.

Object quantifiers arbitrary, modal quantifiers weak.

- **Weak $\Box$:** occurs only in *negative* (antitone) contexts.
- **Weak $\Diamond$:** occurs only in *positive* (monotone) contexts.
- Under these restrictions, the $\mathcal{SL}$ interpolant can be held within $\mathcal{SL}_v$ and therefore retranslates to S5.
- By mid-sequent theorem, interpolants need at most **one free world variable**.
- Closing the free variable with modality gives a modal interpolant.
- World arity is bounded by 1.
- If both quantifiers and modalities are weak, the interpolant can be purely propositional.

Case 3: The prennex Fragment

- Object quantifiers: prenex. Modal quantifiers: arbitrary.
- Mid-sequent is **purely propositional** (world-sort only).
- object quantifiers disappear at mid, leaving propositional structure
- Propositional interpolant obtained, then re-prefixed with object quantifiers.
- Proof-theoretic consequences:
  - Herbrand theorem (object sort).
  - Skolemization admissible (object-only Skolem functions).
  - Second ε-theorem (object sort).
  - Decidability of the fragment.
- Mid is object-quantifier free $\Rightarrow$ propositional interpolant.
- Duality: world-prenex + arbitrary objects also interpolates.

Consequences of EMST

Corollary (Herbrand disjunctions)

For every valid prenex formula in S5 of the form $\exists x_1 \ldots \exists x_n\, \varphi$*, where* φ *is quantifier-free and may contain modalities, there exists a Herbrand disjunction equivalent (in* $\mathcal{SL}$*) to the original formula*

$$\exists \bar{x}\, \varphi(\bar{x}) \;\equiv\; \bigvee_{j=1}^{m} \varphi(\bar{t}_j).$$

Corollary (Skolemization)

existential object quantifiers can be Skolemized with terms that do not need to depend on world variables.

Consequences of EMST

Corollary (Second ε-Theorem)

Every ε-proof of a quantifier-free consequence in the prenex fragment reduces to a Herbrand disjunction (object-side). Thus the second ε-theorem holds for prenex S5.

- **Limits:** classic quantifier-shift failures persist outside the prenex fragment.
- The main novelty is that we only reorganize the proof with respect to object quantifier rules, leaving modal ones untouched.

summery

- embed S5 into a richer **two-sorted** first-order logic $\mathcal{SL}$
- **Separate** world & object terms; ban mixed functions $f(w, x)$.
- Two-sorted separation + cut elimination $\Rightarrow$ mid-sequent theorem.
- **Mid-Sequent Theorem:** middle of a cut-free proof is *propositional in the world sort*.
- **Maehara** on the mid: get propositional interpolant.
- This *does not* automatically give an interpolant in the original modal fragment.
- ask: can interpolants be retranslated back into modal fragment?
- **Retranslate** to the modal fragment; add back only needed (object) prefixes.
- Therefore: **interpolation in two-sorted** $+$ **successful retranslation** $\Rightarrow$ **interpolation in modal S5**.

Conclusion

- We developed a proof-theoretic account of interpolation for S5 via the two-sorted calculus $\mathcal{SL}$.
- Mid-sequent theorems and cut-elimination yield explicit interpolants.
- Retranslation into modal syntax is possible only in restricted cases (often with exponential cost).
- For constant-domain intuitionistic two-sorted systems the same mid-sequent and Herbrand arguments apply. For non-constant domains, separation fails and the argument breaks.
- Open question: *Are $\mathcal{SL}$ interpolants smaller than modal ones?*

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Thank you!

Questions?