Number theory combination: natural density and SMT

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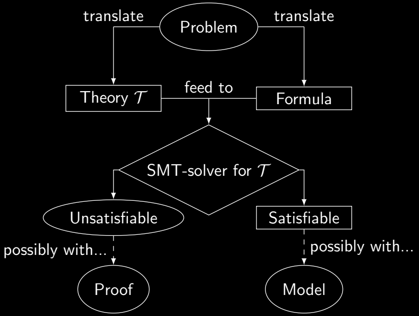
FroCoS 2025

SMT

Satisfiability modulo theories

SMT is an area in automated deduction which seeks to answer whether a formula is satisfied in a (first-order, with equality) theory.



Combination methods

Combining theories is highly non-trivial: methods include Nelson–Oppen, politeness, strong politeness, convexity, flexibility, gentleness (and gentleness), gentleness and computable finite spectra, shininess, etc. and etc.

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A combination method is a theorem that claims more or less the following.

1. If $\mathcal{T}_1$ is decidable, and has **property 1**,
2. and $\mathcal{T}_2$ is decidable, and has **property 2**,
3. and their signatures are disjoint,

then: $\mathcal{T}_1 \oplus \mathcal{T}_2$ is decidable, and the theorem gives a recipe for its algorithm by combining those of $\mathcal{T}_1$ and $\mathcal{T}_2$.

Natural density and spectra

Definitions

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A number is computable when there is an algorithm outputting fractions that converge to it.

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More importantly, we get more tools to study the viability (or not) of theory combination.

Indeed, these developments summarize many of the ideas involved in our recent studies on theory combination, including "Being polite is not enough (and other limits of theory combination)" and "Shininess, Strong Politeness, and Unicorns".

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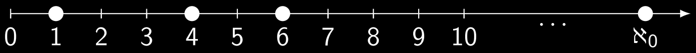

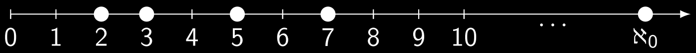

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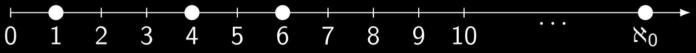

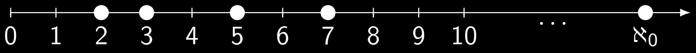

Theorem

By compactness, if $\mu(\mathcal{T}_1), \mu(\mathcal{T}_2) > 0$, then $\mathcal{T}_1$ and $\mathcal{T}_2$ can be combined using Nelson-Oppen.

Shininess

Property 1 in shiny theory combination is shininess, and **property 2** is not needed. Shininess means the least element $\mathbf{minmod}(\phi)$ of $Spec(\mathcal{T}, \phi)$ is computable and finite, and $Spec(\mathcal{T}, \phi) = [\mathbf{minmod}(\phi), \aleph_0]$.

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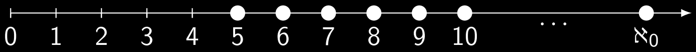

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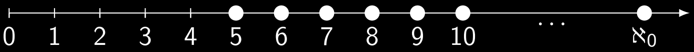

Theorem

If $\mathcal{T}$ is shiny, then $\mu(\mathcal{T}) = 1$.

Gentleness

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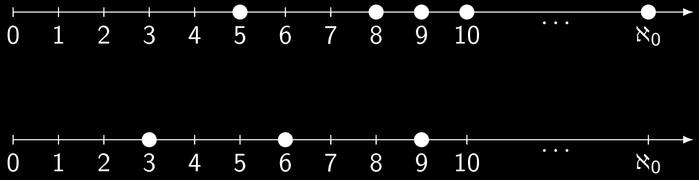

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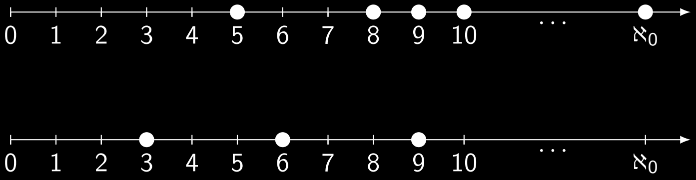

Theorem (a 0 – 1 law)

If $\mathcal{T}$ is gentle, then $\mu(\mathcal{T}) = 0$ or $\mu(\mathcal{T}) = 1$.

Exceptional cases

Computablity of the minimal model function

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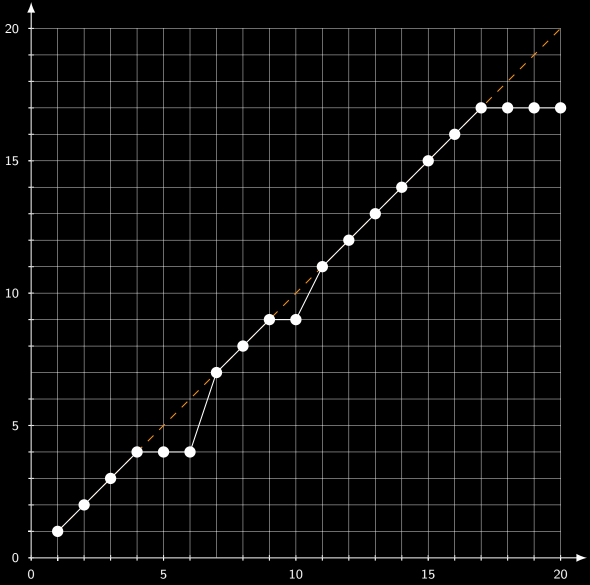
It follows that if $\mathcal{T}$ has a computable minimal model function, then $\mu(\mathcal{T})$ is a computable number. In addition, every computable number is the density of a theory with a computable minimal model function.

Construction



$a_0/b_0 = 4/6$

$a_1/b_1 = 3/4$

$a_2/b_2 = 7/10$

$\vdots$

Density = limit of mediants (Farey sums)

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

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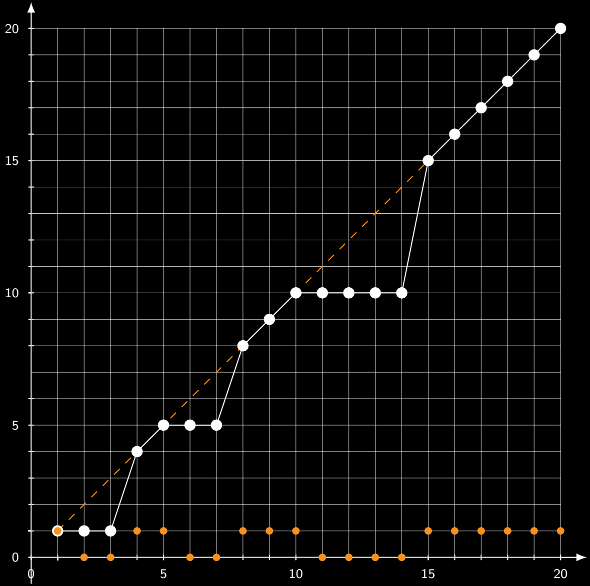
"Being polite is not enough (and other limits of theory combination)"

Every $0 \leq r \leq 1$ is the density of a finitely witnessable theory



$g(1) = 1$

$g(2) = 1$

$g(3) = 0$

$\vdots$

$a_0/b_0 = 1/2$

$a_1/b_1 = 2/3$

$a_2/b_2 = 3/5$

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Theorem

We have, however, that for every set $\{\cdots < a_n < a_{n+1} < \cdots\}$ *of positive density there is a computable sequence* $\{b_n\}_{n\in\mathbb{N}}$ *such that* $a_n \leq b_n$*. So, if* $\mu(\mathcal{T}) > 0$, $\mathcal{T}$ *is finitely witnessable.*

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Conclusion and future work

General picture

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- Many-sorted theories: even more densities in $\mathbb{N}^n$

Thank you!