Shininess, strong politeness, and unicorns

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Theory combination

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- Satisfiability modulo theories (SMT) is the problem of determining whether a formula is satisfiable with respect to a given theory.
- There are plenty of decidable theories used in SMT.
- But in practice, we often need to reason about multiple theories at once.
- Given decision procedures for theories $\mathcal{T}_1$ and $\mathcal{T}_2$, when can we construct a decision procedure for $\mathcal{T}_1 \oplus \mathcal{T}_2$?

Example

- Suppose we want to check the satisfiability of a statement like the following:

$$f(\mathbf{u}[i]) \leq \mathbf{u}[0] \wedge f(\mathbf{u}[0]) = i + 1,$$

where i is an integer, $\mathbf{u}$ is an array, and f is an uninterpreted function.

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  where i is an integer, $\mathbf{u}$ is an array, and f is an uninterpreted function.
- We're combining three theories here: arrays, linear arithmetic, and uninterpreted functions.

Nelson–Oppen method

Nelson and Oppen initiated the study of theory combination by showing that theory combination is possible when $\mathcal{T}_1$ and $\mathcal{T}_2$ are stably infinite.

Definition

A theory $\mathcal{T}$ is *stably infinite* if for every $\mathcal{T}$-satisfiable quantifier-free formula φ, there is a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying φ with $|\mathcal{A}|$ infinite.

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Theorem (Nelson & Oppen 1979, Oppen 1980)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be decidable theories over disjoint signatures. If $\mathcal{T}_1$ and $\mathcal{T}_2$ are stably infinite, then $\mathcal{T}_1 \oplus \mathcal{T}_2$ is decidable.

The theory combination zoo

Several other properties relevant to theory combination:

Combining combination properties

In 2023, Toledo, Zohar, and Barrett began an effort to systematically classify the possible Boolean combinations of theory combination properties:

					Empty		Non-empty		
SI	**SM**	**FW**	**SW**	**CV**	One-sorted	Many-sorted	One-sorted	Many-sorted	$N^{\underline{o}}$
T	T	T	T	T	$\mathcal{T}_{\geq n}$	$(\mathcal{T}_{\geq n})^2$	$(\mathcal{T}_{\geq n})_s$	$((\mathcal{T}_{\geq n})^2)_s$	1
				F	Theorem 5		$(\mathcal{T}_{\geq n})_\vee$	$((\mathcal{T}_{\geq n})^2)_\vee$	2
			F	T	Theorem 3	$\mathcal{T}_{2,3}$	$\mathcal{T}_f$	$(\mathcal{T}_f)_s$	3
				F		Theorem 5	$\mathcal{T}_f^s$	$(\mathcal{T}_{2,3})_\vee$	4
		F	T	T	Theorem 2				5
				F					6
			F	T	$\mathcal{T}_\infty$	$(\mathcal{T}_\infty)^2$	$(\mathcal{T}_\infty)_s$	$((\mathcal{T}_\infty)^2)_s$	7
				F	Theorem 5		$(\mathcal{T}_\infty)_\vee$	$((\mathcal{T}_\infty)^2)_\vee$	8
	F	T	T	T	Theorem 7	Unicorn	Theorem 7	Unicorn	9
				F		Theorem 5			10
			F	T	$\mathcal{T}_{even}^\infty$	$(\mathcal{T}_{even}^\infty)^2$	$(\mathcal{T}_{even}^\infty)_s$	$((\mathcal{T}_{even}^\infty)^2)_s$	11
				F	Theorem 5		$(\mathcal{T}_{even}^\infty)_\vee$	$((\mathcal{T}_{even}^\infty)^2)_\vee$	12
		F	T	T	Theorem 2				13
				F					14
			F	T	$\mathcal{T}_{n,\infty}$	$(\mathcal{T}_{n,\infty})^2$	$(\mathcal{T}_{n,\infty})_s$	$((\mathcal{T}_{n,\infty})^2)_s$	15
				F	Theorem 5		$(\mathcal{T}_{n,\infty})_\vee$	$((\mathcal{T}_{n,\infty})^2)_\vee$	16
F	T	T	T	T	Theorem 1				17
				F					18
			F	T					19
				F					20
		F	T	T	Theorems 1 and 2				21
				F					22
			F	T	Theorem 1				23
				F					24
	F	T	T	T	$\mathcal{T}_{\leq 1}$	$(\mathcal{T}_{\leq 1})^2$	$(\mathcal{T}_{\leq 1})_s$	$((\mathcal{T}_{\leq 1})^2)_s$	25
				F	$\mathcal{T}_{\leq n}$	$(\mathcal{T}_{\leq n})^2$	$(\mathcal{T}_{\leq n})_s$	$((\mathcal{T}_{\leq n})^2)_s$	26
			F	T	Theorem 8	$\mathcal{T}_1^{odd}$	$\mathcal{T}_{odd}^{\neq}$	$(\mathcal{T}_1^{odd})_s$	27
				F	$\mathcal{T}_{m,n}$	$(\mathcal{T}_{m,n})^2$	$(\mathcal{T}_{m,n})_s$	$((\mathcal{T}_{m,n})^2)_s$	28
		F	T	T	Theorem 2				29
				F					30
			F	T	Theorem 6	$\mathcal{T}_1^\infty$	$\mathcal{T}_{1,\infty}^{\neq}$	$(\mathcal{T}_1^\infty)_s$	31
				F		$\mathcal{T}_2^\infty$	$\mathcal{T}_{2,\infty}^{\neq}$	$(\mathcal{T}_2^\infty)_s$	32

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This paper is a continuation of that effort.

Combining combination properties (cont.)

1. Combining Combination Properties: An Analysis of Stable Infiniteness, Convexity, and Politeness
G. V. Toledo, Y. Zohar, and C. Barrett. *CADE 2023*.
2. Combining Finite Combination Properties: Finite Models and Busy Beavers
G. V. Toledo, Y. Zohar, and C. Barrett. *FroCoS 2023*.
3. The nonexistence of unicorns and many-sorted Löwenheim–Skolem theorems
B. Przybocki, G. V. Toledo, Y. Zohar, and C. Barrett. *FM 2024*.
4. Combining Combination Properties: Minimal Models
G. V. Toledo and Y. Zohar. *LPAR 2024*.
5. **Shininess, strong politeness, and unicorns**
B. Przybocki, G. V. Toledo, Y. Zohar. *FroCoS 2025*.

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- We complete the classification of which Boolean combinations of eight properties relevant to theory combination are possible.
- We prove a new decidability result for so-called *shiny* theories.
- We refine a result due to Casal and Rasga regarding the equivalence of shiny and strongly polite theories.
- We study conditions under which strong politeness is equivalent to additive politeness.
- Finally, we resolve an open problem about the relation between finite smoothness and smoothness.

Shininess

Tinelli and Zarba proved a combination theorem that only imposes a requirement on one of the component theories.

Theorem (Tinelli & Zarba 2003)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be decidable theories over disjoint signatures. If $\mathcal{T}_1$ is shiny, then $\mathcal{T}_1 \oplus \mathcal{T}_2$ is decidable.

Definition of shininess

Definition

- The *spectrum* of a formula φ in a theory $\mathcal{T}$, denoted $Spec_{\mathcal{T}}(\varphi)$, is the set of $\kappa \in \mathbb{N} \cup \{\aleph_0\}$ for which there is a $\mathcal{T}$-interpretation $\mathcal{A}$ satisfying φ with $|\mathcal{A}| = \kappa$.

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- A theory $\mathcal{T}$ is *shiny* if each $Spec_{\mathcal{T}}(\varphi)$ is of the form

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- The *minimal model function* of $\mathcal{T}$ is the function minmod such that $\mathrm{minmod}(\varphi) = \min Spec_{\mathcal{T}}(\varphi)$ whenever φ is $\mathcal{T}$-satisfiable.

Strong politeness

Ranise, Ringeissen, and Zarba proved a similar combination theorem:

Theorem (Ranise, Ringeissen, & Zarba 2005, Jovanović & Barrett 2010)

Let $\mathcal{T}_1$ and $\mathcal{T}_2$ be decidable theories over disjoint signatures. If $\mathcal{T}_1$ is strongly polite, then $\mathcal{T}_1 \oplus \mathcal{T}_2$ is decidable.

Definition of politeness

Definition

- Roughly, a theory $\mathcal{T}$ is *finitely witnessable* if given a $\mathcal{T}$-satisfiable quantifier-free formula φ, we can compute an equivalent formula $wit(\varphi)$ satisfied by a $\mathcal{T}$-interpretation in which every element is named by some variable in $wit(\varphi)$.

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- Roughly, a theory is *strongly finitely witnessable* if it satisfies a similar property where we additionally add (dis)equality constraints on the variables in $wit(\varphi)$.

Definition of politeness

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- Roughly, a theory is *strongly finitely witnessable* if it satisfies a similar property where we additionally add (dis)equality constraints on the variables in $wit(\varphi)$.
- A theory is *(strongly) polite* if it is smooth and (strongly) finitely witnessable.

The Casal–Rasga equivalence

Theorem (Casal & Rasga 2018)

A decidable theory is shiny if and only if it is strongly polite.

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Every shiny theory is decidable (and therefore strongly polite).

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Some strongly polite theories are undecidable (and therefore not shiny).

The Casal–Rasga equivalence (cont.)

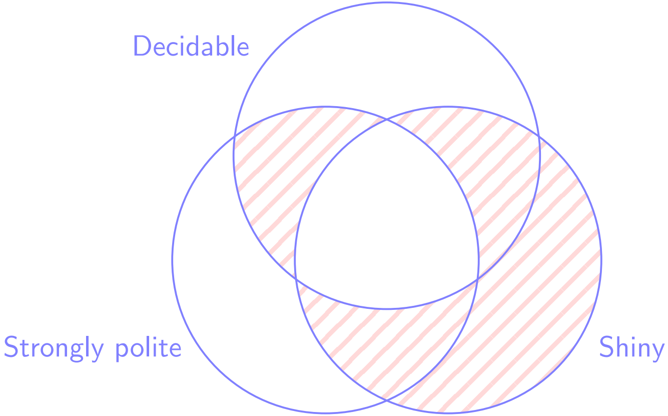

Figure: A Venn diagram summarizing the possible combinations of shininess, strong politeness, and decidability

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- No. Peano arithmetic has a computable minimal model function (let $\mathrm{minmod}(\varphi) = \aleph_0$ for all φ), but it is not decidable.

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Proof sketch.

Suppose $\mathcal{T}$ is shiny. Given φ, let $k = \mathrm{minmod}_{\mathcal{T}}(\varphi)$. By the finite model property, $k < \aleph_0$. Let $x_1, \ldots, x_{k+1}$ be fresh variables. Then, φ is $\mathcal{T}$-satisfiable if and only if

$$\mathrm{minmod}_{\mathcal{T}}\left(\varphi \vee \bigwedge_{1 \leq i < j \leq k+1} x_i \neq x_j\right) = k.$$

□

An undecidable strongly polite theory

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- Fix an undecidable set $S \subset \{n \in \mathbb{N} : n \geq 3\}$.
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- Let

$$cycle_n(x) := f^n(x) = x \land \bigwedge_{1 \leq m < n} f^m(x) \neq x.$$

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- Let

$$cycle_n(x) \coloneqq f^n(x) = x \wedge \bigwedge_{1 \leq m < n} f^m(x) \neq x.$$

- For example, $cycle_6(c_0)$ holds in this model:

$$\begin{tikzcd}
 & c_4 \arrow[r, "f"] & c_5 \arrow[dr, "f"] & \\
c_3 \arrow[ur, "f"] & & & c_0 \arrow[dl, "f"] \\
 & c_2 \arrow[ul, "f"] & c_1 \arrow[l, "f"] &
\end{tikzcd}$$

An undecidable strongly polite theory (cont.)

Theorem (P., Toledo, & Zohar (2025))

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- Let $\mathcal{T}$ be the Σ-theory axiomatized by

$$Ax(\mathcal{T}) = \{(\exists\, x.\ cycle_n(x)) \to \forall\, x.\ f^2(x) \neq x \mid n \in S\}.$$

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- Let $\mathcal{T}$ be the Σ-theory axiomatized by

$$Ax(\mathcal{T}) = \{(\exists\, x.\ cycle_n(x)) \rightarrow \forall\, x.\ f^2(x) \neq x \mid n \in S\}.$$

- $\mathcal{T}$ is strongly polite.
- But $\mathcal{T}$ is undecidable, since

$$cycle_n(x) \wedge f^2(y) = y$$

is $\mathcal{T}$-satisfiable if and only if $n \notin S$.

Unicorns?

In earlier work, Boolean combinations of properties whose possibility had not been determined were called *unicorn theories*:

<table>
<tr><th colspan="2">Empty</th><th colspan="2">Non-empty</th><th></th></tr>
<tr><th>OS</th><th>MS</th><th>OS</th><th>MS</th><th>Nº</th></tr>
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<tr><td>$\mathcal{T}_f^s$</td><td>$(\mathcal{T}_f^s)^2$</td><td>10</td></tr>
<tr><td rowspan="2">[25]</td><td>$(\mathcal{T}_{2,3})_\vee$</td><td>11</td></tr>
<tr><td>$\mathcal{T}_{\mathrm{XVIII}}$</td><td>12</td></tr>
</table>

As a consequence of the previous two results, we can answer all such questions about unicorn theories.

Recap

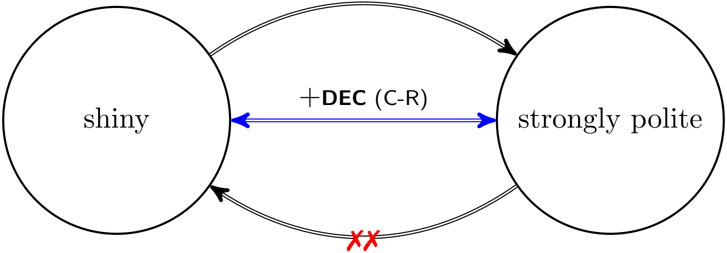

Figure: Summary of the results so far

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- What about the converse?

Additive politeness (cont.)

Theorem (P., Toledo, & Zohar (2025))

Some strongly polite theories are not additively polite.

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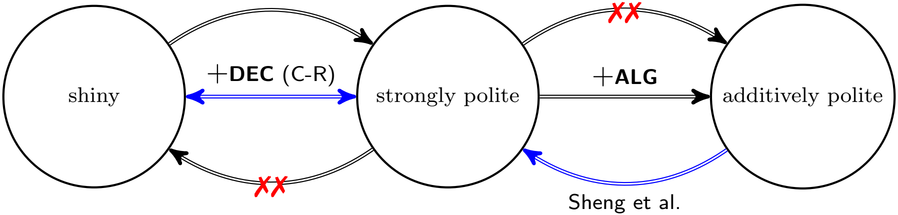
Theorem (P., Toledo, & Zohar (2025))

Every strongly polite theory over an algebraic signature is additively polite.

Summary



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- This completes the classification of which Boolean combinations of eight properties relevant to theory combination are possible.
- Several of the theories that we constructed for this project have been useful for a separate research program we have about impossibility results in theory combination.