# Interpolation for Converse Propositional Dynamic Logic

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TABLEAUX 2025

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# History

- Propositional Dynamic Logic (PDL) and Converse PDL were introduced in 1979 by Fisher and Ladner [Fis97]
- Used to reason about program behaviour
- Open problem: Does (c)PDL have Interpolation?
- Several attempts to solve it: [Lei81], [Bor88], [Kow02]
- [Bor25]: Solution!
  - Only for PDL
  - Complex proof system
- This work
  - For (c)PDL
  - Simpler proof system
  - Simpler correctness argument

# Converse Propositional Dynamic Logic (CPDL)

Let Prop be an infinite set of atomic propositions

Act be an infinite set of atomic programs

$\hat{\ }: \mathrm{Act} \to \mathrm{Act}$ be an involution operator such that $\hat{a} \neq a$ and $\hat{\hat{a}} = a$

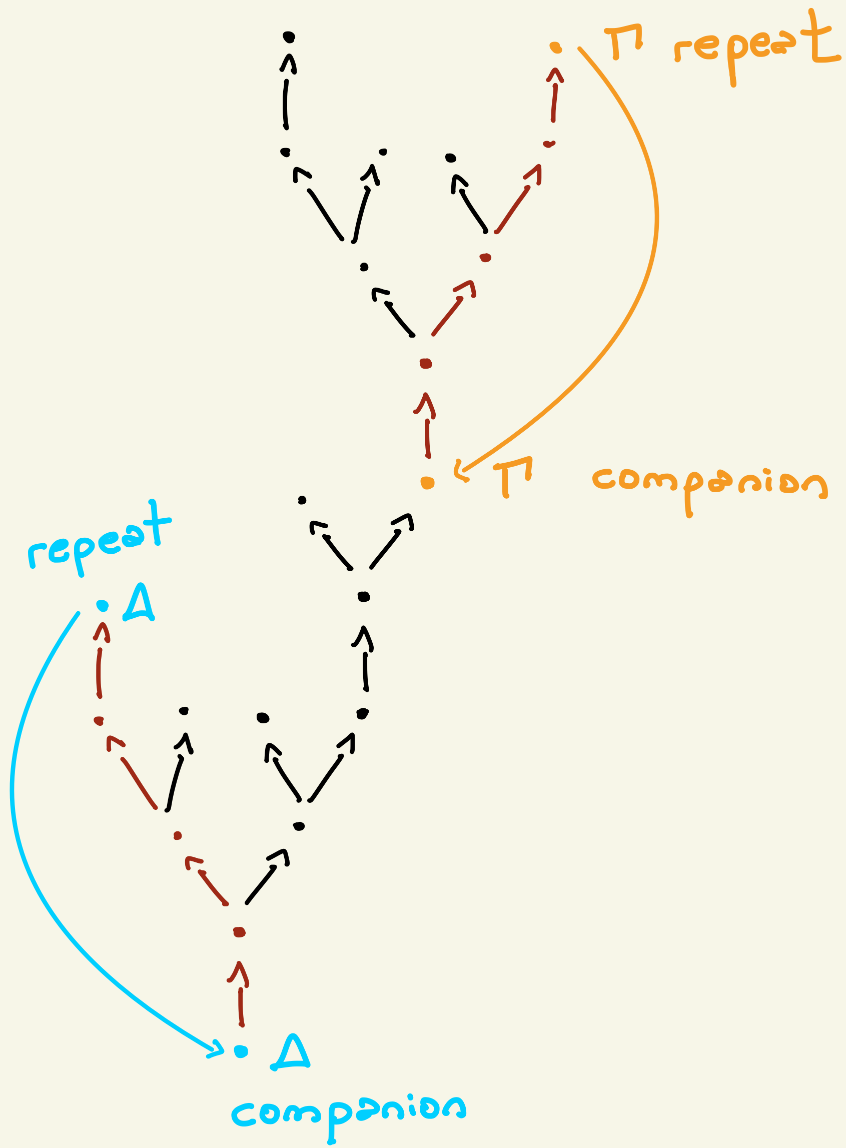
The Sets of Formulas and Programs is defined as

$$\varphi ::= \top \mid \bot \mid p \mid \bar{p} \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle \alpha \rangle \varphi \mid [\alpha]\varphi$$

$$\alpha ::= a \mid \alpha ; \alpha \mid \alpha \cup \alpha \mid \alpha^* \mid \varphi ?$$

where $p \in \mathrm{Prop}$ and $a \in \mathrm{Act}$

# The cyclic tableaux system for CPDL



# The cyclic tableaux system for CPDL

$$\text{Ax1: } \frac{}{\varphi^{o_1}, \overline{\varphi}^{o_2}, \Gamma} \qquad \text{Ax2: } \frac{}{\bot^u, \Gamma} \qquad \wedge\text{: } \frac{\varphi^u, \psi^u, \Gamma}{\varphi \wedge \psi^u, \Gamma} \qquad \vee\text{: } \frac{\varphi^u, \Gamma \qquad \psi^u, \Gamma}{\varphi \vee \psi^u, \Gamma}$$

$$\langle ; \rangle\text{: } \frac{\langle\alpha\rangle\langle\beta\rangle\varphi^o, \Gamma}{\langle\alpha;\beta\rangle\varphi^o, \Gamma} \qquad [;]\text{: } \frac{[\alpha][\beta]\varphi^u, \Gamma}{[\alpha;\beta]\varphi^u, \Gamma} \qquad \langle\cup\rangle\text{: } \frac{\langle\alpha\rangle\varphi^o, \Gamma \qquad \langle\beta\rangle\varphi^o, \Gamma}{\langle\alpha\cup\beta\rangle\varphi^o, \Gamma}$$

$$\langle * \rangle\text{: } \frac{\langle\alpha\rangle\langle\alpha^*\rangle\varphi^o, \Gamma \qquad \varphi^o, \Gamma}{\langle\alpha^*\rangle\varphi^o, \Gamma} \qquad [*]\text{: } \frac{[\alpha][\alpha^*]\varphi^u, \varphi^u, \Gamma}{[\alpha^*]\varphi^u, \Gamma} \qquad [\cup]\text{: } \frac{[\alpha]\varphi^u, [\beta]\varphi^u, \Gamma}{[\alpha\cup\beta]\varphi^u, \Gamma}$$

$$\langle ? \rangle\text{: } \frac{\psi^u, \varphi^o, \Gamma}{\langle\psi?\rangle\varphi^o, \Gamma} \qquad [?]\text{: } \frac{\overline{\psi}^u, \Gamma \qquad \varphi^u, \Gamma}{[\psi?]\varphi^u, \Gamma} \qquad \langle \mathsf{a} \rangle\text{: } \frac{\varphi^f, \Sigma, \langle\breve{a}\rangle\Gamma}{\langle a\rangle\varphi^f, [a]\Sigma, \Gamma}$$

$$\text{acut: } \frac{\Gamma, \varphi^u \qquad \overline{\varphi}^u, \Gamma}{\Gamma} \qquad \text{weak: } \frac{\Gamma}{\varphi^u, \Gamma} \qquad \text{f: } \frac{\varphi^f, \Gamma}{\varphi^u, \Gamma} \qquad \text{u: } \frac{\varphi^u, \Gamma}{\varphi^f, \Gamma}$$

**Fig. 1.** Rules of $\mathsf{CPDL}_f$

# The cyclic tableaux system for CPDL

principal

$$
\AxiomC{$[a^*](p\vee\bot)^u, \langle a^*\rangle\bar{p}^f$}
\UnaryInfC{$p^u, [a][a^*](p\vee\bot)^u, \langle a\rangle\langle a^*\rangle\bar{p}^f$}
\AxiomC{}
\UnaryInfC{$p^u, [a][a^*](p\vee\bot)^u, \bar{p}^f$}
\BinaryInfC{$p^u, [a][a^*](p\vee\bot)^u, \langle a^*\rangle\bar{p}^f$}
\AxiomC{}
\UnaryInfC{$\bot^u, [a][a^*](p\vee\bot)^u, \bar{p}^f$}
\AxiomC{$[a^*](p\vee\bot)^u, \langle a^*\rangle\bar{p}^f$}
\UnaryInfC{$\bot^u, [a][a^*](p\vee\bot)^u, \langle a\rangle\langle a^*\rangle\bar{p}^f$}
\BinaryInfC{$\bot^u, [a][a^*](p\vee\bot)^u, \langle a^*\rangle\bar{p}^f$}
\BinaryInfC{$p\vee\bot^u, [a][a^*](p\vee\bot)^u, \langle a^*\rangle\bar{p}^f$}
\UnaryInfC{$[a^*](p\vee\bot)^u, \langle a^*\rangle\bar{p}^f$}
\DisplayProof
$$

**Successful repeat:**

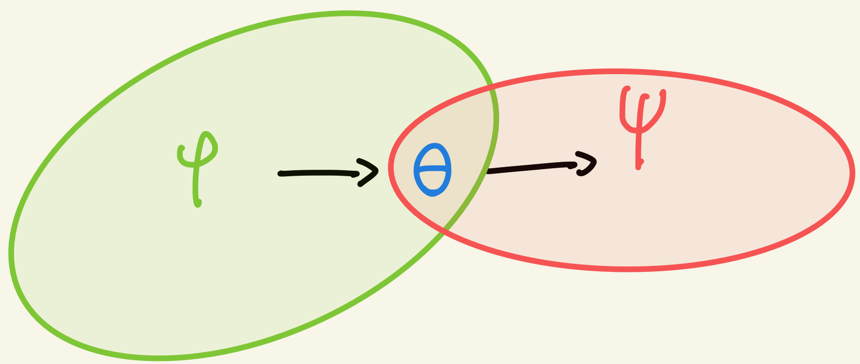
Every sequent in the path companion - repeat is in focus and in at least one node in the path the formula in focus is principal

**Theorem:** Soundness and Completeness of $\vdash$

# Craig Interpolation

We say that a logic $\mathcal{L}$ has Craig Interpolation if for every $\varphi \to \psi \in \mathcal{L}$ there exists a formula $\theta$ such that:

- $\varphi \to \theta \in \mathcal{L}$ and $\theta \to \psi \in \mathcal{L}$
- $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\varphi) \cap \mathrm{Voc}(\psi)$



# CPDL has Craig Interpolation

**Theorem** (Our Contribution)

For all sequents $\Gamma$ and $\Delta$ such that $\vdash \Gamma, \Delta$ there exists a formula $\theta$, such that:

- $\vdash \Gamma, \theta$
- $\vdash \overline{\theta}, \Delta$
- $Voc(\theta) \subseteq Voc(\Gamma) \cap Voc(\Delta)$

**Corolary** CPDL has Craig interpolation

# Maehara's Method

$$\vdash \Pi, \Delta \quad \rightsquigarrow \quad \begin{array}{c} \dfrac{\bar{p} \mid p}{} \qquad \dfrac{\bot \mid q}{} \\ \vdots \\ \Pi \mid \Delta \end{array}$$

# Maehara's Method

$$\vdash \Pi, \Delta \quad \rightsquigarrow \quad \begin{array}{ccc} p & & \top \\ \underline{\bar{p} \mid p} & & \underline{\bot \mid q} \\ & \vdots & \\ & \Pi \mid \Delta & \end{array}$$

# Maehara's Method

$$\vdash \Pi, \Delta \quad \rightsquigarrow \quad \begin{array}{c} \begin{array}{cc} \underline{\overset{p}{\bar{p} \mid p}} & \underline{\overset{\top}{\bot \mid q}} \end{array} \\ \vdots \\ \overset{\theta}{\Pi \mid \Delta} \end{array} \quad \rightsquigarrow \quad \theta$$

# CPDL has Craig Interpolation

Theorem (Our Contribution)

For all sequents $\Gamma$ and $\Delta$ such that $\vdash \Gamma \mid \Delta$ there exists a formula $\theta$, such that:

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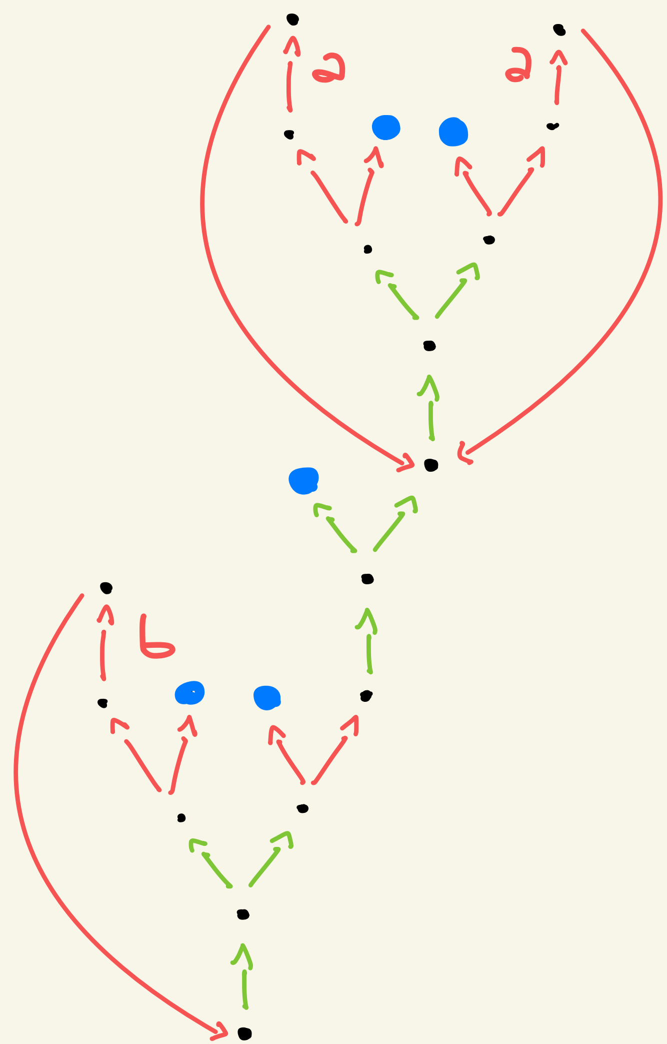
Corolary CPDL has Craig interpolation

## Maehara's with Quasiproofs ([Bor25])



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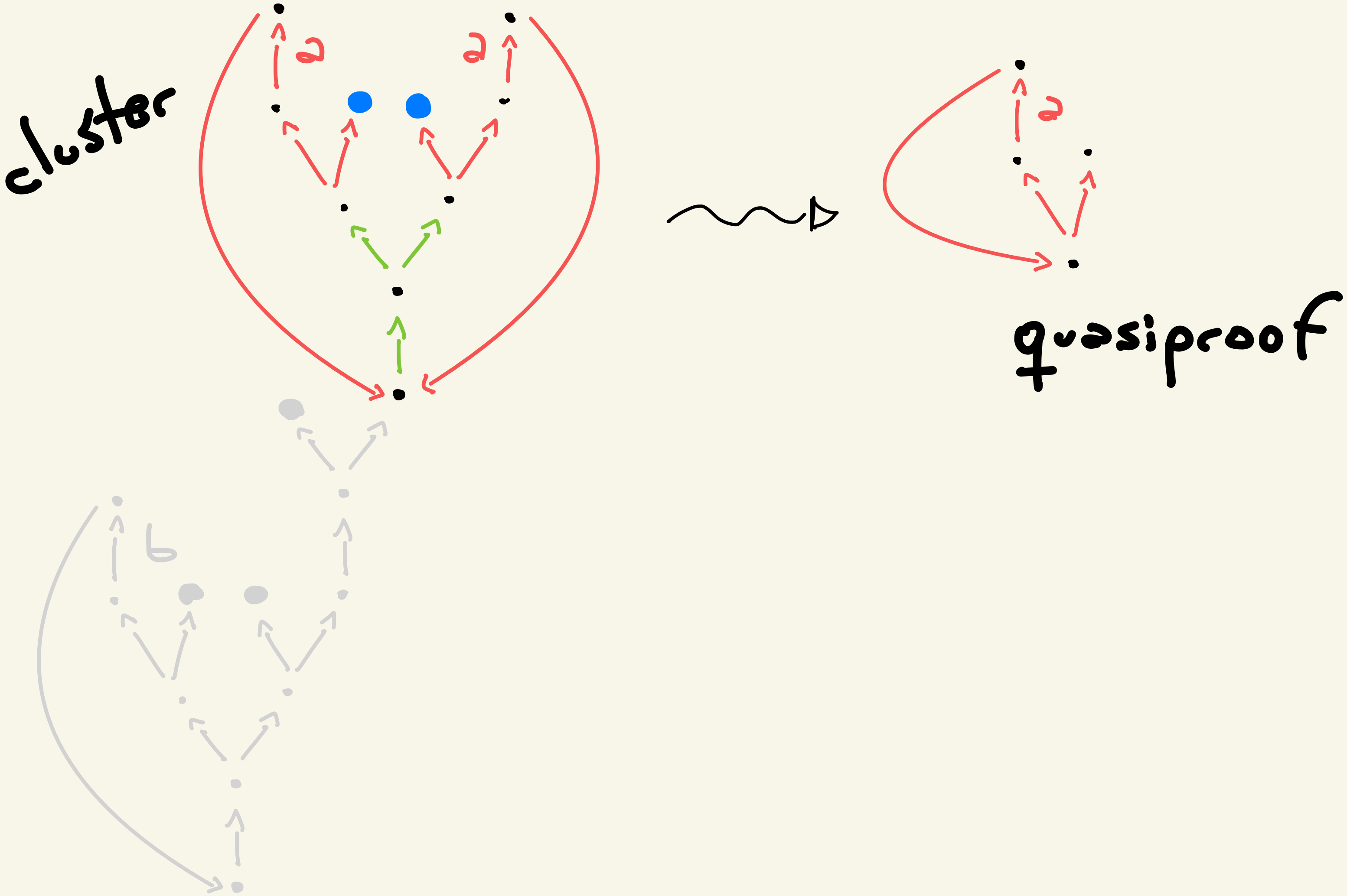

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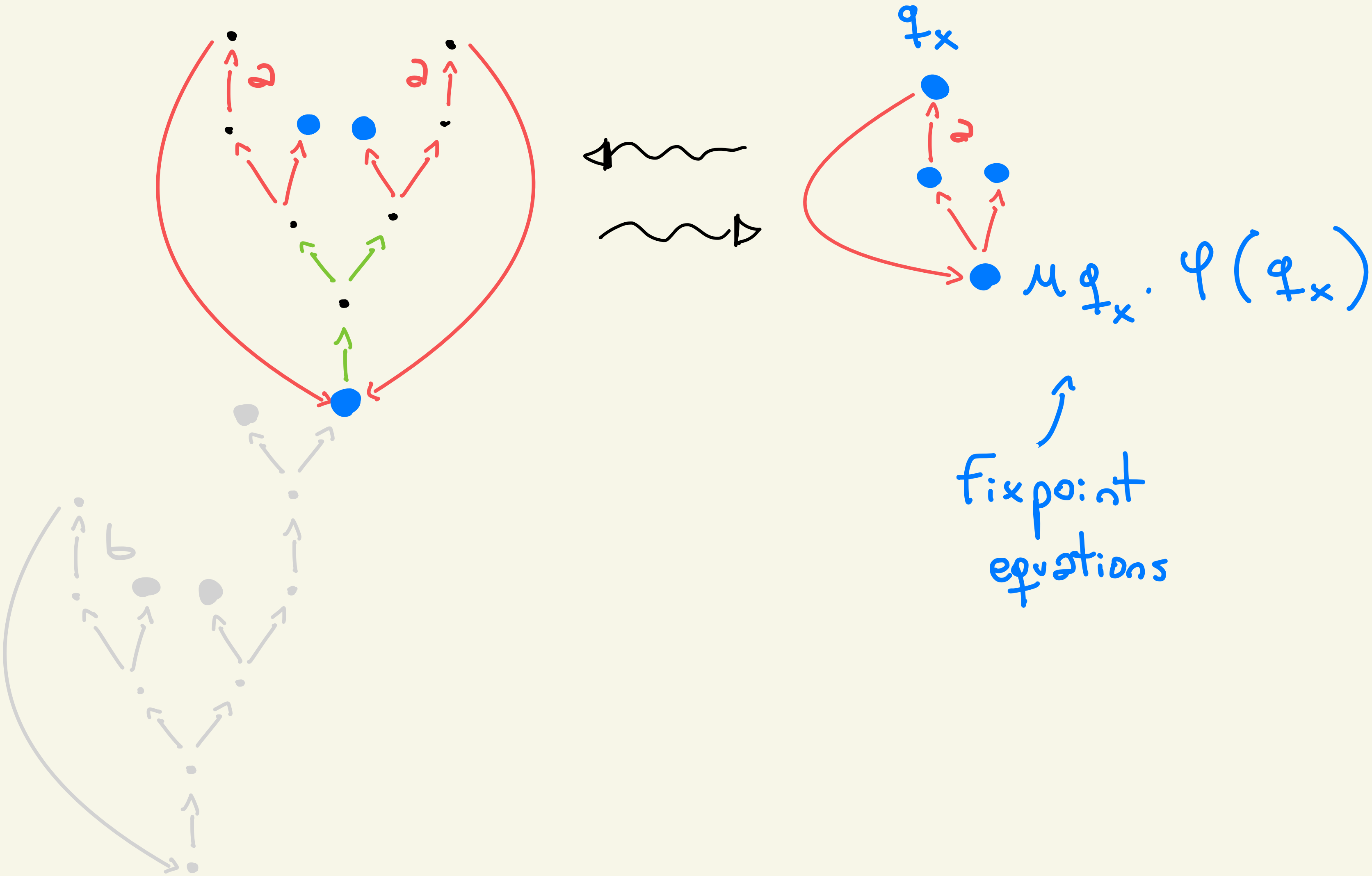

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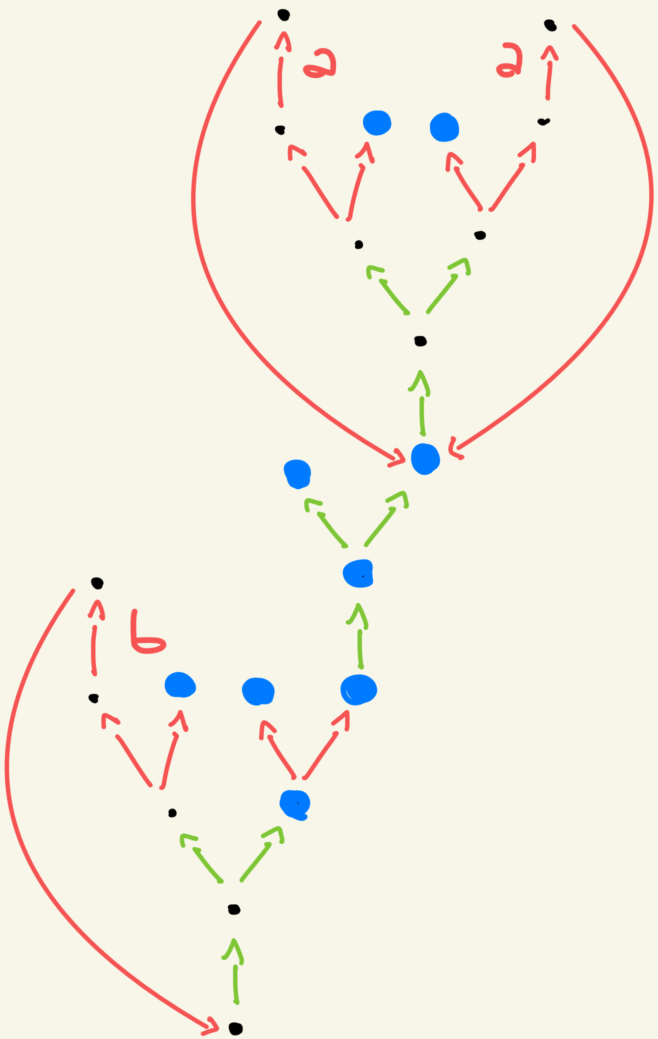

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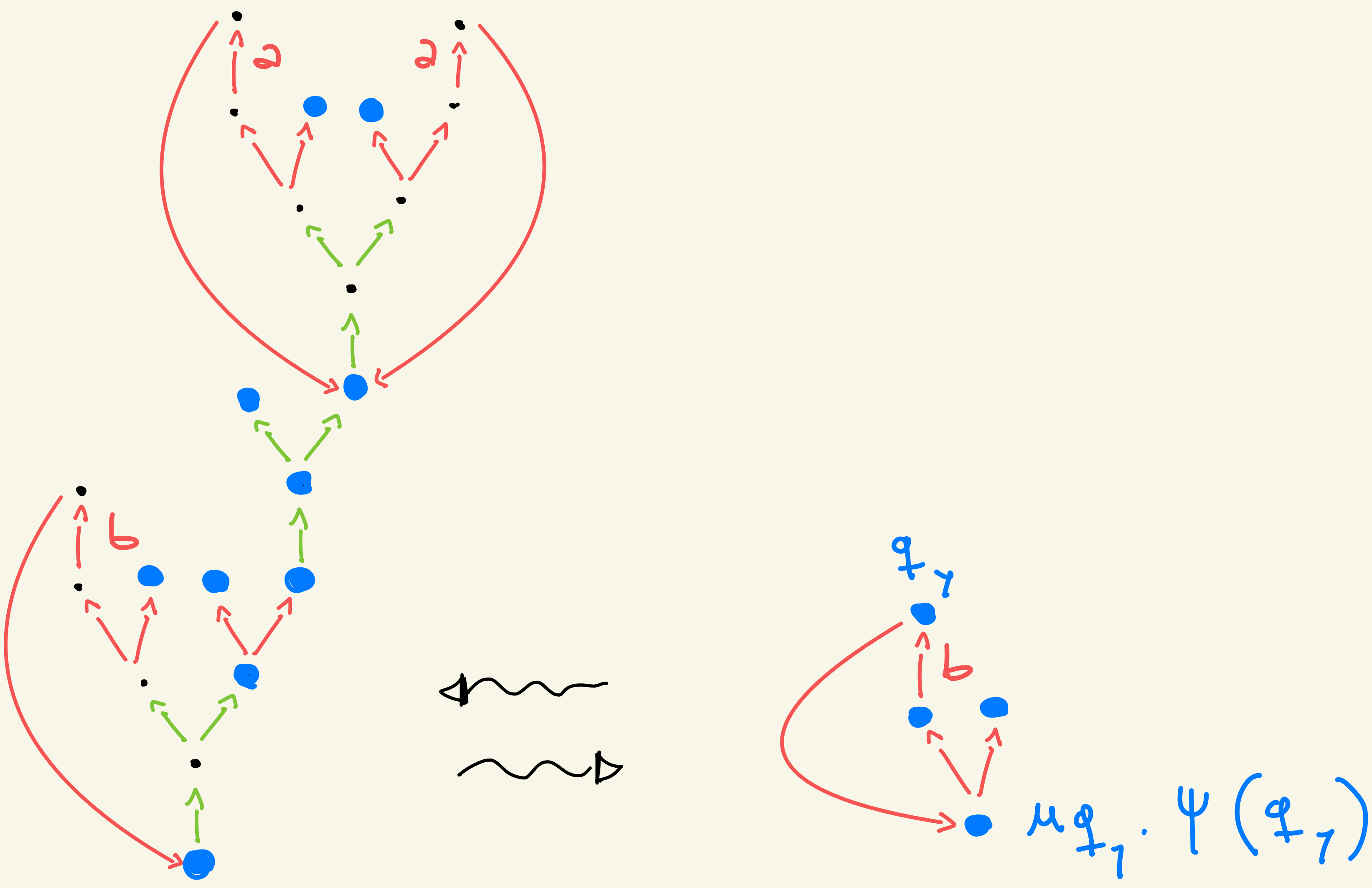
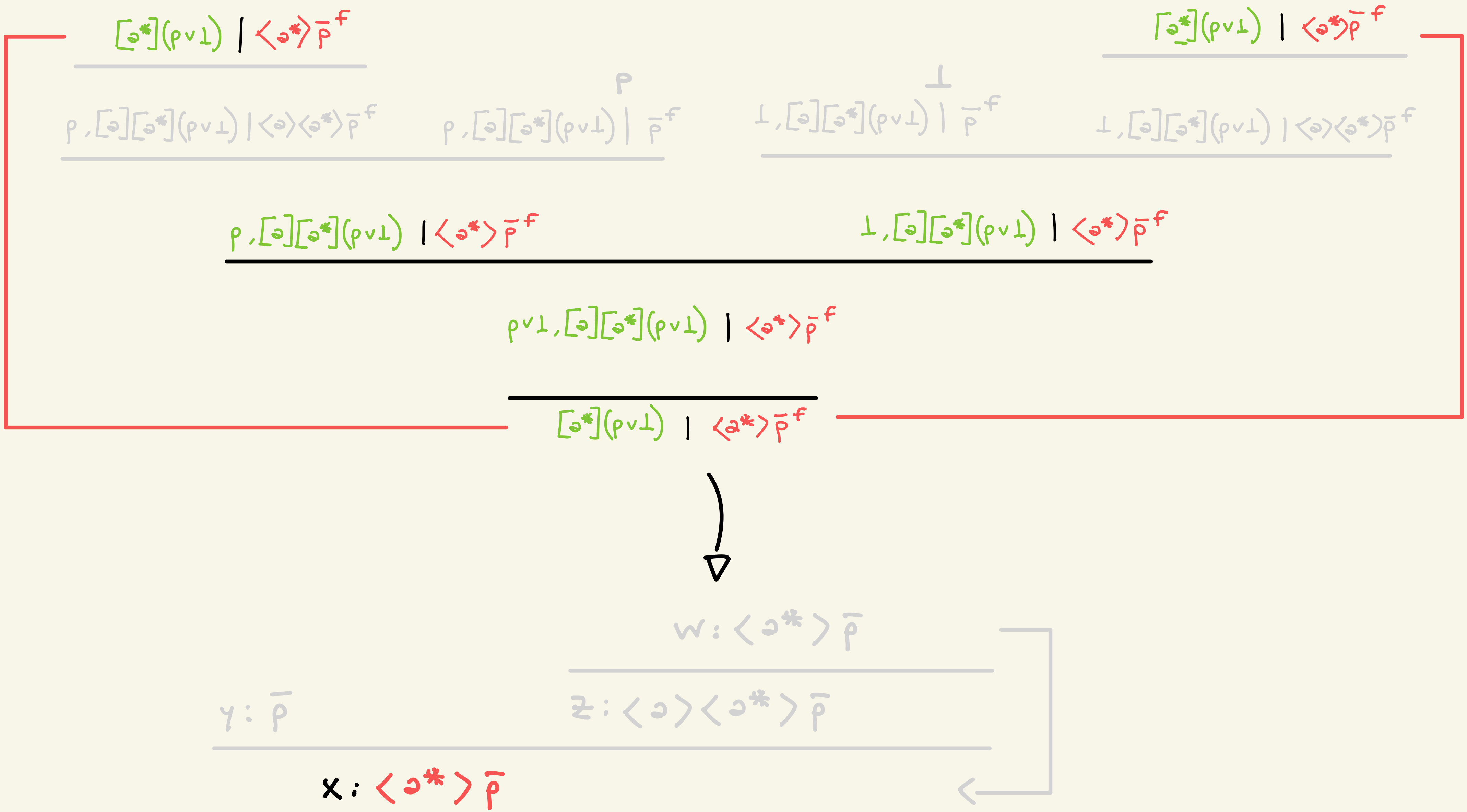


# An Example

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}{p,[a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f}
\qquad
p,[a][a^*](p\vee\bot) \overset{p}{\mid} \bar{p}^f
}{p,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}
\qquad
\dfrac{
\bot,[a][a^*](p\vee\bot) \overset{\bot}{\mid} \bar{p}^f
\qquad
\dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}{\bot,[a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f}
}{\bot,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}
}{p\vee\bot,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}
}{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}
$$

# An Example



$[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f$
$p, [a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f$
$p$
$p, [a][a^*](p\vee\bot) \mid \bar{p}^f$
$\bot$
$\bot, [a][a^*](p\vee\bot) \mid \bar{p}^f$
$\bot, [a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f$
$[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f$
$p, [a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f$
$\bot, [a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f$
$p\vee\bot, [a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f$
$[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f$
$w : \langle a^*\rangle\bar{p}$
$y : \bar{p}$
$z : \langle a\rangle\langle a^*\rangle\bar{p}$
$x : \langle a^*\rangle\bar{p}$

# An Example

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}{p,[a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f} \qquad p,[a][a^*](p\vee\bot) \overset{p}{\mid} \bar{p}^f}{p,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f} \qquad \dfrac{\bot,[a][a^*](p\vee\bot) \overset{\bot}{\mid} \bar{p}^f \qquad \dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}{\bot,[a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f}}{\bot,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}}{p\vee\bot,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}}{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}$$

$$\dfrac{y : \bar{p} \;:\; p\wedge\bot \qquad \dfrac{w : \langle a^*\rangle\bar{p}}{z : \langle a\rangle\langle a^*\rangle\bar{p}}}{x : \langle a^*\rangle\bar{p}}$$

# An Example

$$\dfrac{\dfrac{\dfrac{\dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}{p,[a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f} \quad \dfrac{p}{p,[a][a^*](p\vee\bot) \mid \bar{p}^f}}{p,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f} \quad \dfrac{\dfrac{\bot}{\bot,[a][a^*](p\vee\bot) \mid \bar{p}^f} \quad \dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}{\bot,[a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle\bar{p}^f}}{\bot,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}}{p\vee\bot,[a][a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}}{[a^*](p\vee\bot) \mid \langle a^*\rangle\bar{p}^f}$$

$$\downarrow$$

$$\dfrac{y:\bar{p} : p\wedge\bot \qquad \dfrac{w:\langle a^*\rangle\bar{p}}{z:\langle a\rangle\langle a^*\rangle\bar{p}}}{x:\langle a^*\rangle\bar{p}}$$

# An Example

$$\dfrac{\dfrac{\dfrac{\dfrac{[a^*](p\vee\bot)\mid\langle a^*\rangle\bar{p}^{f}}{p,[a][a^*](p\vee\bot)\mid\langle a\rangle\langle a^*\rangle\bar{p}^{f}}\qquad \dfrac{p}{p,[a][a^*](p\vee\bot)\mid\bar{p}^{f}}}{p,[a][a^*](p\vee\bot)\mid\langle a^*\rangle\bar{p}^{f}}\qquad \dfrac{\dfrac{\bot}{\bot,[a][a^*](p\vee\bot)\mid\bar{p}^{f}}\qquad \dfrac{[a^*](p\vee\bot)\mid\langle a^*\rangle\bar{p}^{f}}{\bot,[a][a^*](p\vee\bot)\mid\langle a\rangle\langle a^*\rangle\bar{p}^{f}}}{\bot,[a][a^*](p\vee\bot)\mid\langle a^*\rangle\bar{p}^{f}}}{p\vee\bot,[a][a^*](p\vee\bot)\mid\langle a^*\rangle\bar{p}^{f}}}{[a^*](p\vee\bot)\mid\langle a^*\rangle\bar{p}^{f}}$$

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# An Example

$$
\dfrac{
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\dfrac{\dfrac{[a^*](p \vee \bot) \mid \langle a^* \rangle \bar{p}^f}{p, [a][a^*](p \vee \bot) \mid \langle a \rangle \langle a^* \rangle \bar{p}^f}
\qquad
\overset{p}{p, [a][a^*](p \vee \bot) \mid \bar{p}^f}}{p, [a][a^*](p \vee \bot) \mid \langle a^* \rangle \bar{p}^f}
\qquad
\dfrac{\overset{\bot}{\bot, [a][a^*](p \vee \bot) \mid \bar{p}^f}
\qquad
\dfrac{[a^*](p \vee \bot) \mid \langle a^* \rangle \bar{p}^f}{\bot, [a][a^*](p \vee \bot) \mid \langle a \rangle \langle a^* \rangle \bar{p}^f}}{\bot, [a][a^*](p \vee \bot) \mid \langle a^* \rangle \bar{p}^f}
}{p \vee \bot, [a][a^*](p \vee \bot) \mid \langle a^* \rangle \bar{p}^f}
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$$

$$\Downarrow$$

$$
\dfrac{y : \bar{p} : p \wedge \bot
\qquad
\dfrac{w : \langle a^* \rangle \bar{p} : q_x}{z : \langle a \rangle \langle a^* \rangle \bar{p} : \langle a \rangle q_x}}{x : \langle a^* \rangle \bar{p} : \mu q_x . \langle a \rangle q_x \vee (p \wedge \bot)}
$$

# An Example

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle \overline{p}^{f}}{p, [a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle \overline{p}^{f}} \qquad \overset{p}{p, [a][a^*](p\vee\bot) \mid \overline{p}^{f}}}{p, [a][a^*](p\vee\bot) \mid \langle a^*\rangle \overline{p}^{f}}
\qquad
\dfrac{\overset{\bot}{\bot, [a][a^*](p\vee\bot) \mid \overline{p}^{f}} \qquad \dfrac{[a^*](p\vee\bot) \mid \langle a^*\rangle \overline{p}^{f}}{\bot, [a][a^*](p\vee\bot) \mid \langle a\rangle\langle a^*\rangle \overline{p}^{f}}}{\bot, [a][a^*](p\vee\bot) \mid \langle a^*\rangle \overline{p}^{f}}
}{p\vee\bot, [a][a^*](p\vee\bot) \mid \langle a^*\rangle \overline{p}^{f}}
}{[a^*](p\vee\bot) \mid \langle a^*\rangle \overline{p}^{f}}
$$

$$
\dfrac{y : \overline{p} : p\wedge\bot \qquad \dfrac{w : \langle a^*\rangle \overline{p} : q_x}{z : \langle a\rangle\langle a^*\rangle \overline{p} : \langle a\rangle q_x}}{x : \langle a^*\rangle \overline{p} : \langle a^*\rangle (p\wedge\bot)}
$$

# Definition of Interpolant

$$\iota_x = \bigvee_{y \in K_{<y}} \langle \alpha_{x,y} \rangle q_y \ \vee \ \psi_x$$

$$\dfrac{y : \bar{p} : p \wedge \bot \qquad \dfrac{w : \langle a^* \rangle \bar{p} : q_x}{z : \langle a \rangle \langle a^* \rangle \bar{p} : \langle a \rangle q_x}}{x : \langle a^* \rangle \bar{p} : \mu q_x . \langle a \rangle q_x \vee (p \wedge \bot)}$$

# Definition of Interpolant

$$\iota_x = \bigvee_{y \in K_{<y}} \langle \alpha_{x,y} \rangle \iota_y \vee \psi_x$$

| **Case** | $\psi_x$ | $\alpha_{x,y}$ |
|---|---|---|
| x is a repeat | $\bot$ | $\top?$ |
| x is an exit | $\theta_{\Psi_x}$ | $-$ |
| x is a companion | $\langle \alpha^*_{z,x} \rangle \psi_z$ | $\alpha^*_{z,x}; \alpha_{z,y}$ |
| x is otherwise of type 1 | $\psi_z$ | $\alpha_{z,y}$ |
| x is of type 2 | $\langle \theta_{\Psi_x}? \rangle \psi_z$ | $\theta_{\Psi_x}?; \alpha_{z,y}$ |
| x is of type 3, not modal | $\bigvee \{ \psi_z \mid x \lessdot_{\mathsf{Q}} z \}$ | $\bigcup \{ \alpha_{z,y} \mid x \lessdot_{\mathsf{Q}} z, y \in \mathsf{K}_{<z} \}$ |
| x is of type 3, modal | $\langle a \rangle \psi_z$ | $a; \alpha_{z,y}$ |

# Correctness of Interpolant

Fix $\Gamma, \Delta$ such that $\vdash \Gamma \mid \Delta$.

We must show that

ⓐ $\vdash \Gamma \mid \theta$

ⓑ $\vdash \Delta \mid \overline{\theta}$

ⓒ $\mathrm{Voc}(\theta) \subseteq \mathrm{Voc}(\Gamma) \cap \mathrm{Voc}(\Delta)$

$\| \vdash \Gamma \mid \theta$

## Lemma

For all $x \in Q$, for all $y \in K_{<x}$ and all $t \in R_x$ we have

(1) $\vdash \Gamma_t \mid \psi_x^u$

(2) $A_y \vdash \Gamma_t \mid \langle \alpha_{x,y} \rangle \varphi_y^f$ with $A_y = \{ (\Gamma_s \mid \varphi_y^f) \mid s \in R_y \}$

**Lemma** For all $x \in Q$, for all $y \in K_{<x}$ and all $t \in R_x$ we have

(1) $\vdash \Gamma_t \mid \psi_x^u$

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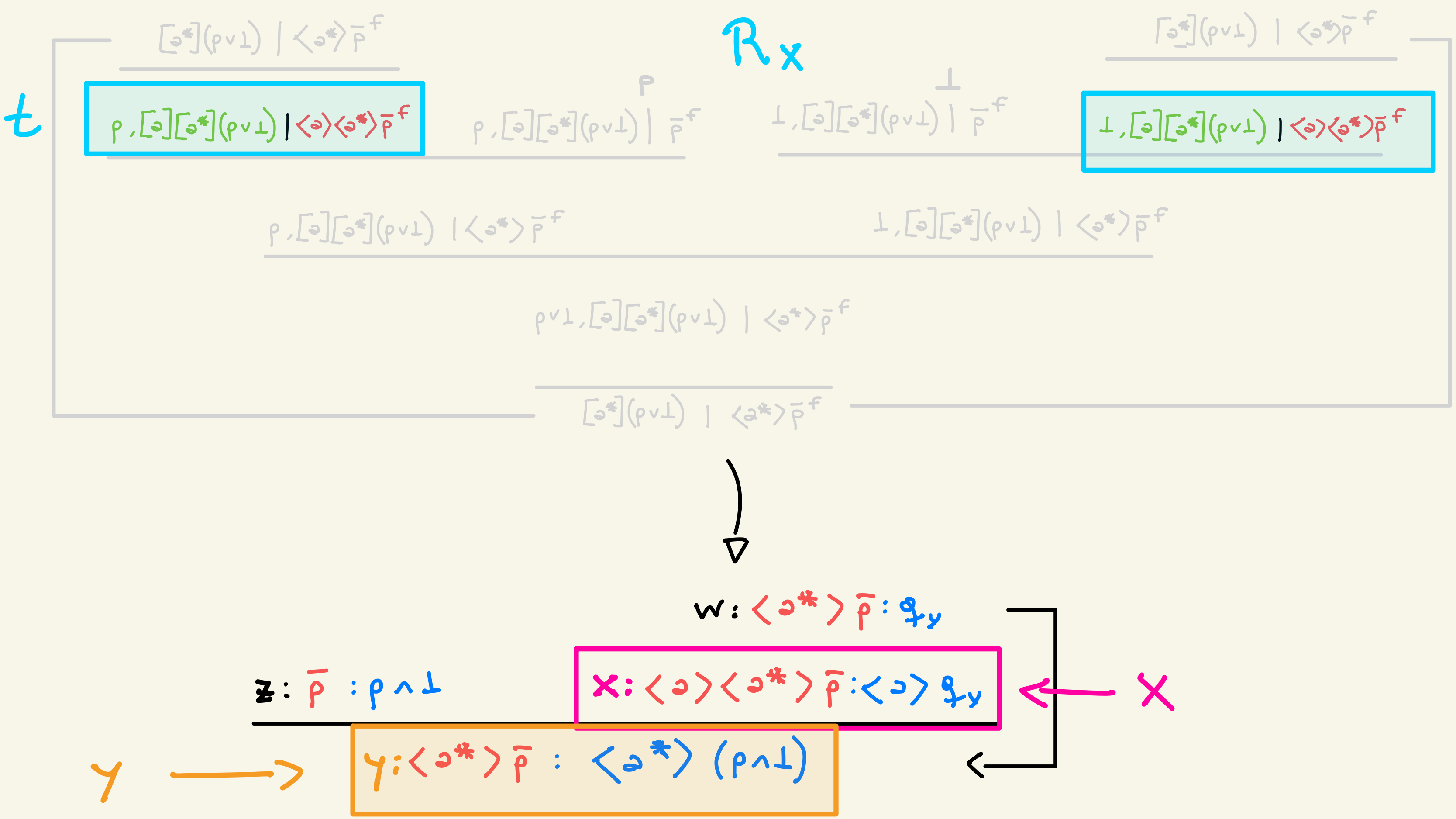

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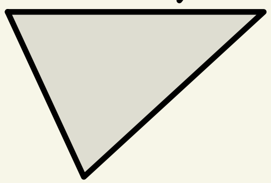
**Proof** Induction on $x$. Case: $x$ is a companion.

$x$ must have a successor $z$. Recall $\alpha_{x,y} = \alpha_{z,x}^* ; \alpha_{z,y}$.

By IH on $z$ we have the following proofs for all $y \in K_{<x}$:

$[\Gamma_s \mid q_y^f]_{s \in R_y}$



$\Gamma_t \mid \langle \alpha_{z,y} \rangle q_y^f$

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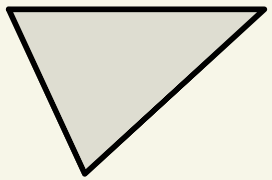

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$x$ must have a successor $z$. Recall $\alpha_{x,y} = \alpha_{z,x}^* ; \alpha_{z,y}$.

By IH on $z$ we have the following proofs for all $y \in K_{<x}$:

$$\dfrac{\begin{array}{c}[\Gamma_s \mid \varphi_y^f]_{s \in R_y} \\ \vdots \\ \Gamma_t \mid \langle \alpha_{z,y} \rangle \varphi_y^f\end{array} \qquad \begin{array}{c}[\Gamma_s \mid \langle \alpha_{z,x}^* \rangle \langle \alpha_{z,y} \rangle \varphi_y^f]_{s \in R_x} \\ \vdots \\ \Gamma_t \mid \langle \alpha_{z,x} \rangle \langle \alpha_{z,x}^* \rangle \langle \alpha_{z,y} \rangle \varphi_y^f\end{array}}{\Gamma_t \mid \langle \alpha_{z,x}^* \rangle \langle \alpha_{z,y} \rangle \varphi_y^f}$$

$\| \vdash \bar{\theta} \mid \Delta$

<u>Lemma</u> For all $x \in Q$ there is a proof $B_x \vdash^c \overline{L_x}^{\,v} \mid \Delta_x$

with $B_x = \{ (\overline{F_y}^{\,v} \mid \Delta_y) \mid y \in K_{<x} \}$

Lemma For all $x \in Q$ there is a proof $\mathcal{B}_x \vdash^c \overline{L_x}^{\,u} \mid \Delta_x$

with $\mathcal{B}_x = \{(\overline{q_y}^{\,u} \mid \Delta_y) \mid y \in K_{<x}\}$

Proof Induction on $x$. Case $x$ is companion. Let $z$ be the child of $x$.

Then by IH:

$$\begin{array}{c} \mathcal{B}_x \cup \{\overline{q_x}^{\,u} \mid \Delta_x\} \\ \vdots \; \pi_z \\ \overline{L_z} \mid \Delta_z \end{array}$$

**Lemma** For all $x \in Q$ there is a proof $B_x \vdash^c \overline{L_x}^u \mid \Delta_x$

with $B_x = \{ (\overline{q_y}^u \mid \Delta_y) \mid y \in K_{<x} \}$

**Proof** Induction on $x$. Case $x$ is companion. Let $z$ be the child of $x$.

Then by IH: $\pi_z$
$$\begin{array}{c} B_x \cup \{ \overline{q_x}^u \mid \Delta_x \} \\ \vdots \\ \overline{L_z} \mid \Delta_z \end{array}$$
. Then:

$L_z[L_x/q_x] \equiv L_x$

$$\dfrac{\begin{array}{c} B_x \cup \{ \overline{L_x}^u \mid \Delta_x \} \\ \vdots\ \pi_z[L_x] \\ \overline{L_z[L_x/q_x]}^u \mid \Delta_z \end{array} \qquad \begin{array}{c} \vdots\ \rho \\ \overline{L_z[L_x/q_x]}^u, \overline{L_x} \mid \Delta_z \end{array}}{\overline{L_x}^u \mid \Delta_z}\ \mathrm{cut}^{\ell}$$

**Lemma** For all $x \in Q$ there is a proof $B_x \vdash^{c} \overline{L_x}^{\circ} \mid \Delta_x$

with $B_x = \{ (\overline{q_y}^{\circ} \mid \Delta_y) \mid y \in K_{<x} \}$

**Proof** Induction on $x$. Case $x$ is companion. Let $z$ be the child of $x$.

Then by IH:
$$\begin{array}{c} B_x \cup \{ \overline{q_x}^{\circ} \mid \Delta_x \} \\ \vdots\ \pi_z \\ \overline{L_z} \mid \Delta_z \end{array}$$
. Then:

$$L_z[L_x/q_x] \equiv L_x$$

$$\dfrac{\begin{array}{c} B_x \cup \{ \overline{L_x}^{\circ} \mid \Delta_x \} \\ \vdots\ \pi_z[L_x] \\ \overline{L_z[L_x/q_x]}^{\circ} \mid \Delta_x \end{array} \qquad \begin{array}{c} \vdots\ \rho \\ \overline{L_z[L_x/q_x]}^{\circ}, \overline{L_x} \mid \Delta_x \end{array}}{\overline{L_x}^{\circ} \mid \Delta_x} \, \mathrm{cut}^{\ell}$$

# Contributions

- Sound and Complete tableau system with analytic cut for (C)PDL
- Simpler tableau system for (C)PDL than [Bor25]
  - Simpler handling of unguarded formulas
  - Based on the system for the two-way modal $\mu$-calculus [KV25]
- A proof of Interpolation for CPDL
- A simpler proof of Interpolation for PDL
  - An amalgamation of the ideas from [KV25] and [Bor25]

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