Base-extension Semantics for Intuitionistic Modal Logics

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Goals for this talk

Give an overview of Base-extension Semantics for Intuitionistic Propositional Logic.

Introduce Intuitionistic Modal Logics à la Simpson.

Discuss how to adapt the semantics for IPL to IMLs.

What is the idea of Base-extension Semantics?

Use bases of “atomic rules” to justify inference of atomic formulae (atoms).

Define a satisfaction relation of formulae in bases to give meaning via an inductive definition of validity of formulae.

Showing soundness of the semantics amounts to showing that every inference figure of NJ corresponds to a proof in terms of the definitions of the formulae.

Showing completeness of the semantics amounts to building a special base that simulates proofs in NJ.

The system NJ

$$\frac{}{\top}\ \top_{\mathsf{I}} \qquad\qquad \frac{\bot}{\phi}\ \bot_{\mathsf{E}}$$

$$\frac{\begin{matrix}[\phi]\\ \psi\end{matrix}}{\phi \supset \psi}\ \supset_{\mathsf{I}} \qquad\qquad \frac{\phi \supset \psi \quad \phi}{\psi}\ \supset_{\mathsf{E}}$$

$$\frac{\phi \quad \psi}{\phi \wedge \psi}\ \wedge_{\mathsf{I}} \qquad\qquad \frac{\phi \wedge \psi}{\phi}\ \wedge_{1\mathsf{E}} \quad \frac{\phi \wedge \psi}{\psi}\ \wedge_{2\mathsf{E}}$$

$$\frac{\phi}{\phi \vee \psi}\ \vee_{1\mathsf{I}} \quad \frac{\psi}{\phi \vee \psi}\ \vee_{2\mathsf{I}} \qquad\qquad \frac{\phi \vee \psi \quad \begin{matrix}[\phi]\\ \chi\end{matrix} \quad \begin{matrix}[\psi]\\ \chi\end{matrix}}{\chi}\ \vee_{\mathsf{E}}$$

What might a general inference figure look like?

$$\dfrac{\begin{matrix} [\Gamma_1] & & [\Gamma_n] \\ \gamma_1 & \cdots & \gamma_n \end{matrix}}{\phi}$$

Derivability in NJ

If we suppose a formula $\phi \in \Gamma$, then it is clear that $\Gamma \vdash_{\mathrm{NJ}} \phi$.

Consider the general inference figure

$$\frac{\begin{array}{ccc} [\Gamma_1] & & [\Gamma_n] \\ \gamma_1 & \cdots & \gamma_n \end{array}}{\phi}$$

If $\Delta, \Gamma_i \vdash_{\mathrm{NJ}} \gamma_i$ for all i then $\Delta \vdash_{\mathrm{NJ}} \phi$.

Atomic rules

$$\dfrac{\begin{matrix}[P_1] & & [P_n] \\ p_1 & \ldots & p_n\end{matrix}}{q}$$

Linearly we write this as:

$$(P_1 \Rightarrow p_1, \ldots, P_n \Rightarrow p_n) \Rightarrow q$$

Example of atomic rules

Example of atomic rules

$$\frac{}{\text{Tableaux 2025 is happening in Reykjavik}}$$

Example of atomic rules

$$\frac{\text{Freyja is a cat} \qquad \text{Freyja is female}}{\text{Freyja is a læða}}$$

$$\frac{\text{Freyja is a læða}}{\text{Freyja is a cat}}$$

$$\frac{\text{Freyja is a læða}}{\text{Freyja is female}}$$

Example of a hypothesis discharging rule

$$\frac{\text{Var1 is a byte} \qquad \begin{matrix}[\text{Var1 has value 0}] \\ \phi\end{matrix} \qquad \cdots \qquad \begin{matrix}[\text{Var1 has value 255}] \\ \phi\end{matrix}}{\phi}$$

Atomic derivability in a base $\mathscr{B}$

A base $\mathscr{B}$ is a set of atomic rules.

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A base $\mathscr{B}$ is a set of atomic rules.

(Ref) $S \vdash_{\mathscr{B}} p$ if $p \in S$

(App) If there is a rule $((P_1 \Rightarrow p_1, \ldots, P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}$ such that $S, P_i \vdash_{\mathscr{B}} p_i$ then $S \vdash_{\mathscr{B}} q$.

Example derivations using atomic rules

Example

Let $\mathscr{B} = \{(\Rightarrow a), ((\Rightarrow a), (\Rightarrow b) \Rightarrow c)\}$

$$\dfrac{\dfrac{}{b \vdash_{\mathscr{B}} b}\,\text{Ref} \qquad \dfrac{}{b \vdash_{\mathscr{B}} a} \Rightarrow a}{b \vdash_{\mathscr{B}} c}\,(\Rightarrow a), (\Rightarrow b) \Rightarrow c$$

Example

Let $\mathscr{B} = \{(\Rightarrow c), ((\Rightarrow a), (b \Rightarrow c) \Rightarrow d), ((\Rightarrow a), (\Rightarrow d) \Rightarrow e), ((a \Rightarrow e) \Rightarrow f)\}$

$$\dfrac{\dfrac{\dfrac{}{a \vdash_{\mathscr{B}} a}\,\mathrm{Ref} \qquad \dfrac{\dfrac{}{a \vdash_{\mathscr{B}} a}\,\mathrm{Ref} \qquad \dfrac{}{a, b \vdash_{\mathscr{B}} c}\,\Rightarrow c}{a \vdash_{\mathscr{B}} d}\,(\Rightarrow a), (b \Rightarrow c) \Rightarrow d}{a \vdash_{\mathscr{B}} e}\,(\Rightarrow a), (\Rightarrow d) \Rightarrow e}{\vdash_{\mathscr{B}} f}\,(a \Rightarrow e) \Rightarrow f$$

BeS for IPL: Summary

(At) $\Vdash_{\mathscr{B}} p$ iff $\vdash_{\mathscr{B}} p$.

($\wedge$) $\Vdash_{\mathscr{B}} \phi \wedge \psi$ iff $\Vdash_{\mathscr{B}} \phi$ and $\Vdash_{\mathscr{B}} \psi$.

($\vee$) $\Vdash_{\mathscr{B}} \phi \vee \psi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, p, if $\phi \Vdash_{\mathscr{C}} p$ and $\psi \Vdash_{\mathscr{C}} p$ then $\Vdash_{\mathscr{C}} p$.

($\supset$) $\Vdash_{\mathscr{B}} \phi \supset \psi$ iff $\phi \Vdash_{\mathscr{B}} \psi$.

($\bot$) $\Vdash_{\mathscr{B}} \bot$ iff $\Vdash_{\mathscr{B}} p$ for all p.

($\top$) $\Vdash_{\mathscr{B}} \top$ iff always.

(Inf) $\Gamma \Vdash_{\mathscr{B}} \phi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$ and $\gamma \in \Gamma$ if $\Vdash_{\mathscr{C}} \gamma$ then $\Vdash_{\mathscr{C}} \phi$.

(Val) $\Gamma \Vdash \phi$ iff $\Gamma \Vdash_{\mathscr{B}} \phi$ for all $\mathscr{B}$.

Soundness and Completeness

Theorem (Soundness)

If $\Gamma \vdash_{\mathrm{NJ}} \phi$ *then* $\Gamma \Vdash \phi$

- $\Gamma \vdash_{\mathrm{NJ}} \phi$ means we have an NJ derivation of ϕ from Γ.
- If $\phi \in \Gamma$ then clearly $\Gamma \Vdash \phi$.
- We thus argue by the inductive definition of an NJ derivation.
- Assume by IH the hypothesis of each rule of NJ is valid. Show the conclusion holds.

Theorem (Completeness)

If $\Gamma \Vdash \phi$ *then* $\Gamma \vdash_{\mathrm{NJ}} \phi$

- To prove completeness we construct a special base $\mathscr{N}$ whose rules simulate NJ.
- To simulate, we mean assign a unique atom to each subformula of the sequent $(\Gamma : \phi)$.
- Since bases do not contain schemas, every rule must be simulated for every subformula in every position of every rule.

Completeness continued

We let $(\cdot)^\flat$ represent this assignment and $(\cdot)^\natural$ be it's right inverse.

- $L \Vdash_{\mathscr{B}} p$ if and only if $L \vdash_{\mathscr{B}} p$.
- For any $\mathscr{B} \supseteq \mathscr{N}$, $\Vdash_{\mathscr{B}} \phi$ if and only if $\Vdash_{\mathscr{B}} \phi^\flat$.
- If $L \vdash_{\mathscr{N}} p$ then $L^\natural \vdash_{\mathrm{NJ}} p^\natural$.

Proof.

(1) Start by noting that $\Gamma \Vdash \phi$ implies that $\Gamma \Vdash_{\mathscr{N}} \phi$.

(2) By the second point above: $\Gamma^\flat \Vdash_{\mathscr{N}} \phi^\flat$.

(3) By the first point above $\Gamma^\flat \vdash_{\mathscr{N}} \phi^\flat$.

(4) Finally, we have by the third point above that $(\Gamma^\flat)^\natural \vdash_{\mathrm{NJ}} (\phi^\flat)^\natural$, that is, $\Gamma \vdash_{\mathrm{NJ}} \phi$.

$\square$

Introduction to Intuitionistic Modal Logics

Modal formulae are defined by the grammar
$\phi, \psi ::= p \in \mathbb{A} \mid \phi \wedge \psi \mid \phi \vee \psi \mid \phi \supset \psi \mid \Box\phi \mid \Diamond\phi \mid \bot \mid \top$.

We consider a new class of object called labelled formulae. We write these as ϕ^x. The label is interpreted as the "locale" the formula holds at. The set of all such "locales" is written as $\mathbb{W}$.

We allow for a binary relation on labels called a relational assumption. If the labels x and y are related, we write this as xRy.

An atomic formula is now either a labelled propositional atom or a relational assumption.

A modal sequent is an object $(\Gamma : \phi^x)$ where Γ is a set of labelled formulae and relational assumptions and ϕ^x is a labelled formula.

An extended sequent is either a modal sequent or an ordered pair $(\emptyset, xRy)$.

Frame conditions

Axiom Schema	Label	Name	Relational property
$\Diamond\top$	γ_D	Seriality	$\forall x.\, \exists y.\, xRy$
$\Box\phi \supset \phi$	γ_T	Reflexivity	$\forall x.\, xRx$
$\phi \supset \Box\Diamond\phi$	γ_B	Symmetry	$\forall x.\, \forall y.\, xRy \Rightarrow yRx$
$\Box\phi \supset \Box\Box\phi$	γ_4	Transitivity	$\forall x.\, \forall y.\, \forall z.\, xRy \;\&\; yRz \Rightarrow xRz$
$\Diamond\phi \supset \Box\Diamond\phi$	γ_5	Euclidean	$\forall x.\, \forall y.\, \forall z.\, xRy \;\&\; xRz \Rightarrow yRz$
$\Diamond\Box\phi \supset \Box\Diamond\phi$	γ_2	Directed	$\forall x.\, \forall y.\, \forall z.\, xRy \;\&\; yRz \Rightarrow \exists w.\, yRw \;\&\; zRw$

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$\Box\phi \supset \Box\Box\phi$	γ_4	Transitivity	$\forall x.\, \forall y.\, \forall z.\, xRy \;\&\; yRz \Rightarrow xRz$
$\Diamond\phi \supset \Box\Diamond\phi$	γ_5	Euclidean	$\forall x.\, \forall y.\, \forall z.\, xRy \;\&\; xRz \Rightarrow yRz$
$\Diamond\Box\phi \supset \Box\Diamond\phi$	γ_2	Directed	$\forall x.\, \forall y.\, \forall z.\, xRy \;\&\; yRz \Rightarrow \exists w.\, yRw \;\&\; zRw$

In what follows we fix an arbitrary set of frame conditions

$$\gamma \subseteq \{\gamma_D, \gamma_T, \gamma_B, \gamma_4, \gamma_5, \gamma_2\}$$

Doing so amounts to fixing a particular modal logic, as we shall see.

The system $\mathrm{N}_{\Box\Diamond}(\gamma)$

$$\frac{}{\top^x}\ \top_{\mathsf{I}} \qquad \frac{\bot^x}{\phi^y}\ \bot_{\mathsf{E}}$$

$$\frac{\begin{array}{c}[\phi^x]\\ \psi^x\end{array}}{(\phi \supset \psi)^x}\ \supset_{\mathsf{I}} \qquad \frac{(\phi \supset \psi)^x \quad \phi^x}{\psi^x}\ \supset_{\mathsf{E}}$$

$$\frac{\phi^x \quad \psi^x}{(\phi \wedge \psi)^x}\ \wedge_{\mathsf{I}} \qquad \frac{(\phi \wedge \psi)^x}{\phi^x}\ \wedge_{1\mathsf{E}} \quad \frac{(\phi \wedge \psi)^x}{\psi^x}\ \wedge_{2\mathsf{E}}$$

$$\frac{\phi^x}{(\phi \vee \psi)^x}\ \vee_{1\mathsf{I}} \quad \frac{\psi^x}{(\phi \vee \psi)^x}\ \vee_{2\mathsf{I}} \qquad \frac{(\phi \vee \psi)^x \quad \begin{array}{c}[\phi^x]\\ \chi^y\end{array} \quad \begin{array}{c}[\psi^x]\\ \chi^y\end{array}}{\chi^y}\ \vee_{\mathsf{E}}$$

The system $\mathrm{N}_{\Box\Diamond}(\gamma)$ continued

$$\frac{\begin{array}{c}[xRy] \\ \phi^{y}\end{array}}{(\Box\phi)^{x}}\ \Box_{\mathrm{I}}^{*}$$

$$\frac{(\Box\phi)^{x} \quad xRy}{\phi^{y}}\ \Box_{\mathrm{E}}$$

$$\frac{\phi^{y} \quad xRy}{(\Diamond\phi)^{x}}\ \Diamond_{\mathrm{I}}$$

$$\frac{(\Diamond\phi)^{x} \quad \begin{array}{c}[\phi^{y}]\,[xRy] \\ \psi^{z}\end{array}}{\psi^{z}}\ \Diamond_{\mathrm{E}}^{**}$$

$*$ The label *y* is different to *x* and the labels of any open assumptions.

$**$ The label *y* is different to *x* and *z* and the labels of any open assumptions.

The system $\mathrm{N}_{\Box\Diamond}(\gamma)$ continued

$$\dfrac{\begin{matrix}[xRy] \\ \phi^z\end{matrix}}{\phi^z}\;(R_D)^*$$

$$\dfrac{\begin{matrix}[xRx] \\ \phi^y\end{matrix}}{\phi^y}\;(R_T)$$

$$\dfrac{xRy \quad \begin{matrix}[yRx] \\ \phi^z\end{matrix}}{\phi^z}\;(R_B)$$

$$\dfrac{xRy \quad yRz \quad \begin{matrix}[xRz] \\ \phi^w\end{matrix}}{\phi^w}\;(R_4)$$

$$\dfrac{xRy \quad xRz \quad \begin{matrix}[yRz] \\ \phi^w\end{matrix}}{\phi^w}\;(R_5)$$

$$\dfrac{xRy \quad xRz \quad \begin{matrix}[yRw]\,[zRw] \\ \phi^v\end{matrix}}{\phi^v}\;(R_2)^{**}$$

$*$ The label y is different to x and the labels of any open assumptions.

$**$ The label w is different to v, x, y, z and the labels of any open assumptions.

What might a general inference figure look like now?

$$\dfrac{\begin{matrix}[\Theta_1] & & [\Theta_n] \\ \theta_1 & \dots & \theta_n\end{matrix}}{\phi}$$

What might a general inference figure look like now?

$$\frac{\begin{array}{ccc}[\Theta_1] & & [\Theta_n] \\ \theta_1 & \ldots & \theta_n\end{array}}{\phi}$$

Note that Θ_i are now allowed to contain relational assumptions as well. If any Θ_i is empty, then the corresponding θ_i is allowed to be a relational assumption. If the rule has no premises, then ϕ is allowed to be a relational assumption.

Derivability in $\mathrm{N}_{\Box\Diamond}(\gamma)$

Need a graph $\mathcal{G} = (X, \mathfrak{R})$ where $X \subset \mathbb{W}$ and $\mathfrak{R}$ is a set of relational assumptions on X.

γ's relational properties are a set of conditions imposed on elements of $\mathfrak{R}$.

If we suppose a formula $\phi^x \in \Theta$, then $\Theta \vdash^{\gamma}_{\mathcal{G}} \phi^x$.

Similarly, if we assume $xRy \in \Theta$, then $\Theta \vdash^{\gamma}_{\mathcal{G}} xRy$.

Consider the general inference figure of $\mathrm{N}_{\Box\Diamond}(\gamma)$

$$\frac{\begin{matrix}[\Theta_1] & & [\Theta_n] \\ \theta_1 & \ldots & \theta_n\end{matrix}}{\phi}$$

If $\Delta, \Theta_i \vdash^{\gamma}_{\mathcal{G}} \theta_i$ for all i then $\Delta \vdash^{\gamma}_{\mathcal{G}} \phi$.

Derivability in $\mathrm{N}_{\Box\Diamond}(\gamma)$ continued

We consider a special graph, the trivial graph τ defined as $\tau = (x, \emptyset)$.

A labelled formula ϕ^x is called a theorem of $\mathrm{N}_{\Box\Diamond}(\gamma)$ if $\vdash_\tau^\gamma \phi^x$ holds. We write this as $\vdash^\gamma \phi^x$.

In what follows, we restrict our attention to derivations over the trivial graph only as that suffices for our result. However, what follows readily generalises to the case of non-trivial graphs too.

Atomic rules, Reloaded

As before, atomic rules take the following shape:

$$\dfrac{\begin{matrix}[P_1] & & [P_n] \\ p_1 & \ldots & p_n\end{matrix}}{q}$$

Linearly we write this as:

$$(P_1 \Rightarrow p_1, \ldots, P_n \Rightarrow p_n) \Rightarrow q$$

Atomic derivability in base $\mathscr{B}$

(Ref) $S, p \vdash^{\gamma}_{\mathscr{B}} p$

(App) If $((P_1 \Rightarrow p_1, \ldots, P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}$ and $S, P_i \vdash^{\gamma}_{\mathscr{B}} p_i$ for each i, then $S \vdash^{\gamma}_{\mathscr{B}} q$

Atomic derivability in base $\mathscr{B}$

(Ref) $S, p \vdash^{\gamma}_{\mathscr{B}} p$

(App) If $((P_1 \Rightarrow p_1, \ldots, P_n \Rightarrow p_n) \Rightarrow q) \in \mathscr{B}$ and $S, P_i \vdash^{\gamma}_{\mathscr{B}} p_i$ for each i, then $S \vdash^{\gamma}_{\mathscr{B}} q$

(D) If $\gamma_D \in \gamma$ and there exists a y such that $S, xRy \vdash^{\gamma}_{\mathscr{B}} p^z$, then $S \vdash^{\gamma}_{\mathscr{B}} p^z$

(T) If $\gamma_T \in \gamma$ and $S, xRx \vdash^{\gamma}_{\mathscr{B}} p^y$, then $S \vdash^{\gamma}_{\mathscr{B}} p^y$

(B) If $\gamma_B \in \gamma$, $S \vdash^{\gamma}_{\mathscr{B}} xRy$ and $S, yRx \vdash^{\gamma}_{\mathscr{B}} p^z$, then $S \vdash^{\gamma}_{\mathscr{B}} p^z$

(4) If $\gamma_4 \in \gamma$, $S \vdash^{\gamma}_{\mathscr{B}} xRy$, $S \vdash^{\gamma}_{\mathscr{B}} yRz$, and $S, xRz \vdash^{\gamma}_{\mathscr{B}} p^w$, then $S \vdash^{\gamma}_{\mathscr{B}} p^w$

(5) If $\gamma_5 \in \gamma$, $S \vdash^{\gamma}_{\mathscr{B}} xRy$, $S \vdash^{\gamma}_{\mathscr{B}} xRz$, and $S, yRz \vdash^{\gamma}_{\mathscr{B}} p^w$, then $S \vdash^{\gamma}_{\mathscr{B}} p^w$

(2) If $\gamma_2 \in \gamma$, $S \vdash^{\gamma}_{\mathscr{B}} xRy$, $S \vdash^{\gamma}_{\mathscr{B}} xRz$, and there exists a w such that $S, yRw, zRw \vdash^{\gamma}_{\mathscr{B}} p^v$, then $S \vdash^{\gamma}_{\mathscr{B}} p^v$.

BeS for IMLs: Non-modal cases

(At) $\Vdash^{\gamma}_{\mathscr{B}} p^x$ iff $\vdash^{\gamma}_{\mathscr{B}} p^x$.

(Rel) $\Vdash^{\gamma}_{\mathscr{B}} xRy$ iff $\vdash^{\gamma}_{\mathscr{B}} xRy$.

($\wedge$) $\Vdash^{\gamma}_{\mathscr{B}} (\phi \wedge \psi)^x$ iff $\Vdash^{\gamma}_{\mathscr{B}} \phi^x$ and $\Vdash^{\gamma}_{\mathscr{B}} \psi^x$.

($\vee$) $\Vdash^{\gamma}_{\mathscr{B}} (\phi \vee \psi)^x$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, p^z, if $\phi^x \Vdash^{\gamma}_{\mathscr{C}} p^z$ and $\psi^x \Vdash^{\gamma}_{\mathscr{C}} p^z$ then $\Vdash^{\gamma}_{\mathscr{C}} p^z$.

($\supset$) $\Vdash^{\gamma}_{\mathscr{B}} (\phi \supset \psi)^x$ iff $\phi^x \Vdash^{\gamma}_{\mathscr{B}} \psi^x$.

($\bot$) $\Vdash^{\gamma}_{\mathscr{B}} \bot^x$ iff $\Vdash^{\gamma}_{\mathscr{B}} p^z$ for all p^z.

($\top$) $\Vdash^{\gamma}_{\mathscr{B}} \top^x$ iff always.

(Inf) $\Gamma \Vdash^{\gamma}_{\mathscr{B}} \phi$ iff for all $\mathscr{C} \supseteq \mathscr{B}$ and $\gamma \in \Gamma$ if $\Vdash^{\gamma}_{\mathscr{C}} \gamma$ then $\Vdash^{\gamma}_{\mathscr{C}} \phi$.

(Val) $\Gamma \Vdash^{\gamma} \phi$ iff $\Gamma \Vdash^{\gamma}_{\mathscr{B}} \phi$ for all $\mathscr{B}$.

BeS for IMLs: Modal cases

($\Box$) $\Vdash^{\gamma}_{\mathscr{B}} (\Box\phi)^{x}$ iff $xRy \Vdash^{\gamma}_{\mathscr{B}} \phi^{y}$, for all y.

($\Diamond$) $\Vdash^{\gamma}_{\mathscr{B}} (\Diamond\phi)^{x}$ iff for all $\mathscr{C} \supseteq \mathscr{B}$, p^{z}, if $xRy, \phi^{y} \Vdash^{\gamma}_{\mathscr{C}} p^{z}$ then $\Vdash^{\gamma}_{\mathscr{C}} p^{z}$.

Soundness and Completeness

Theorem (Soundness)

If $\Gamma \vdash^{\gamma} \phi$ *then* $\Gamma \Vdash^{\gamma} \phi$

- The proof follows exactly as before.
- Now must show explicitly also each modal rule is also sound if the corresponding frame condition is in γ.
- Care must be taken with showing the soundness of the rules $\Box_{\mathsf{I}}$, $\Diamond_{\mathsf{E}}$, R_D and R_2.

Why must care be taken?

Consider the rule $\Box_{\mathsf{I}}$. It says that

$$\cfrac{\begin{array}{c}[xRy] \\ \phi^y\end{array}}{(\Box\phi)^x}\ \Box_{\mathsf{I}}$$

with the caveat y is different to x and the labels of any open assumption in a derivation. We have to make sure this side condition is adhered to.

Thus, in this case of the soundness proof, we suppose that $\Gamma, xRy \Vdash^{\gamma} \phi^y$ where y is a label different to x and the labels of any element of Γ, and try to show $\Gamma \Vdash^{\gamma} (\Box\phi)^x$.

Theorem (Completeness)

If $\Gamma \Vdash^{\gamma} \phi$ *then* $\Gamma \vdash^{\gamma} \phi$

- We again argue as before, constructing a simulation base $\mathscr{N}$.
- Similar care must be taken when constructing $\mathscr{N}$ to ensure that the instances of the rules $\Box_{\mathsf{I}}$, $\Diamond_{\mathsf{E}}$, R_D and R_2 in $\mathscr{N}$ satisfy the correct conditions.
- Furthermore, we have to make sure that if an atomic derivation holds due to a modal case of the derivability relation, that this correctly maps to an application of the corresponding rule in $\mathrm{N}_{\Box\Diamond}(\gamma)$.

Thank you!

Thank you for listening!

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