# Justification Logic for Intuitionistic Modal Logic

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September 2025

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- $\Box A \rightsquigarrow t{:}A$, where $t$ is a proof of $A$, or explicit knowledge of $A$.
- Read this as $t$ justifies $A$.

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- Allow monotonicity of proofs

$$s_i{:}A \to s_1 + s_2{:}A$$

where $i \in \{1, 2\}$.

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- Other justification logics include:
  - Logics for S5 cube [Artemov et al., Brezhnev, Brünnler et al.]
  - Geach/Scott-Lemmon logics [Fitting]
  - Gödel-Löb logic [Shamkanov, Fitting]
  - Non-normal modal logics [Rohani and Studer]

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$$
\text{can}\ \frac{A \text{ is an axiom instance}}{c_n{:}\ldots{:}c_1{:}A}
$$

- Add other proof operators and axioms if required.

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This mirrors the external modal view:

$$\mathsf{k}\ \frac{A_1, \ldots, A_n \Rightarrow B}{\Box A_1, \ldots, \Box A_n \Rightarrow \Box B}$$

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- As a corollary, justification logic can emulate necessitation

$$\text{nec}\,\frac{\vdash A}{\vdash \Box A}$$

### Corollary

*If* JL $\vdash A$, *then there exists a proof term* $t$ *such that*

$$\mathsf{LP} \vdash t{:}A$$

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- Realisation can be done proof-theoretically or semantically.

[Artemov, Fitting]

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- Method was expanded to hypersequent and nested sequent calculi.

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- Method was expanded to hypersequent and nested sequent calculi.

  [Artemov et. al, Brünnler et. al]
- Briefly outline the method for constructing a realisation with nested sequents. [Goetschi and Kuznets]

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- E.g. $\Gamma\{\ \} = A, \big[B, [\{\ \}, [D]]\big]$. Then

$$\Gamma\{C\} = A, \big[B, [C, [D]]\big]$$

# Nested system nK

$$\mathsf{id}\,\frac{}{\Gamma\{a,\bar{a}\}} \qquad \wedge\,\frac{\Gamma\{A\} \quad \Gamma\{B\}}{\Gamma\{A\wedge B\}} \qquad \vee\,\frac{\Gamma\{A,B\}}{\Gamma\{A\vee B\}} \qquad \Box\,\frac{\Gamma\{[A]\}}{\Gamma\{\Box A\}}$$

$$\mathsf{k}\,\frac{\Gamma\{[A,\Delta]\}}{\Gamma\{\Diamond A,[\Delta]\}}$$

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**Theorem (Soundness and Completeness)**

$$\mathsf{K} \vdash A \iff \mathsf{nK} \vdash A$$

(Proved using a cut-elimination argument)

$$\mathsf{cut}\ \frac{\Gamma\{A\} \quad \Gamma\{\bar{A}\}}{\Gamma\{\varnothing\}}$$

[Brünnler, Poggiolesi]

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**Theorem**

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- Proceed by induction on the height of the proof of $A$.

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- Difficulty here is dealing with nesting where rules are applied within brackets.

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- Consider the following:

$$\wedge \frac{p, [q_1] \quad p, [q_2]}{p, [q_1 \wedge q_2]}$$

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  - $\mathsf{JK} \vdash (p \vee t_1{:}q_1) \to (p \vee t_2{:}q_2) \to (p \vee t(t_1, t_2){:}(q_1 \wedge q_2))$

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$$\mathsf{mp}\ \frac{x{:}A \to t \cdot x{:}A \quad (x{:}A \to x{:}A) \to B}{?}$$

where $t$ is a proof of $A \to A$.

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- Smallest intuitionistic modal logic with $\Diamond$ is *constructive* modal logic CK. [Bellin et al.]

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- CK (and some extensions) have Gentzen-style proof theory [Bierman and de Paiva]

# Intuitionistic diamonds

- Idea: Make diamond explicit with satisfiers.

$$\Diamond A \rightsquigarrow \mu{:}A$$

- $\mu{:}A$ read as $\mu$ satisfies $A$, or $\mu$ is a model of $A$.

[Kuznets, Marin, Straßburger]

- Language $\mathcal{L}_{\mathsf{J}}$:

$$A ::= \bot \mid p \mid (A \land A) \mid (A \lor A) \mid (A \to A) \mid t{:}A \mid \mu{:}A$$

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- Each proof term replaced with a $\Box$.
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- Achieved through the following:

### Definition (Forgetful projection)

The forgetful projection is a map

$$(\cdot)^{\mathsf{f}} : \mathcal{L}_{\mathsf{J}} \to \mathcal{L}_{\Box}$$

inductively defined as follows:

$$\begin{aligned} \bot^{\mathsf{f}} &:= \bot \\ p^{\mathsf{f}} &:= p \\ (A * B)^{\mathsf{f}} &:= (A^{\mathsf{f}} * B^{\mathsf{f}}) \text{ where } * \in \{\wedge, \vee, \to\} \end{aligned} \qquad \begin{aligned} (t{:}A)^{\mathsf{f}} &:= \Box A^{\mathsf{f}} \\ (\mu{:}A)^{\mathsf{f}} &:= \Diamond A^{\mathsf{f}} \end{aligned}$$

# Establishing correspondence: JIK to IK

Result is formally stated as follows:

**Theorem**

$\mathsf{JIK} \vdash A \Rightarrow \mathsf{IK} \vdash A^{\mathsf{f}}$.

**Proof.**

Apply forgetful projection on axioms of JIK to see we get theorems of IK.
For example:

$$\mathsf{jk}_4 : ((\mu{:}A \to t{:}B) \to \mu \triangleright t{:}(A \to B))^{\mathsf{f}} = (\Diamond A \to \Box B) \to \Box(A \to B)$$

$\square$

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### Definition (Realisation)

A realisation is a map $(\cdot)^r : \mathcal{L}_\Box \to \mathcal{L}_\mathsf{J}$ such that $(A^r)^\mathsf{f} = A$ for each $A \in \mathcal{L}_\Box$.

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Correspondence is formally stated as:

**Theorem (Marin and P.)**

*There exists a realisation function $r$ such that*

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- Adapt the method in the classical case.

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- Provides a treatment to make $\Diamond$ explicit for most intuitionistic modal logics.
- Method for realisation can be applied to other nested sequent systems for intuitionistic modal logics.

  [Arisaka et al., Kuznets and Straßburger]

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- Exploring connections to arithmetic.

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\mathsf{K}, \mathsf{S4} \arrow[r] \arrow[d, "\neg\neg"'] & \mathsf{PA} \arrow[d, "\neg\neg"] \\
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- Understanding $\Diamond$ and satisfiers in classical logic.

$$\mathsf{IK} + \mathsf{LEM} = \mathsf{K} \quad \rightsquigarrow \quad \mathsf{JIK} + \mathsf{LEM} \stackrel{?}{=} \mathsf{JK}$$

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- Semantics for JIK.
- Proof theory of JIK