A sequent calculus perspective on proof-theoretic semantics

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Introduction

- **Proof-theoretic semantics (PtS)** aims to provide accounts of logic in which notions of *proof*, and not of *truth*, are considered the basic units of semantic analysis.
- Since it is usually defined using proofs in **natural deduction** and relies on proof-theoretic harmony for semantic definitions, traditional PtS is better suited to deal with specific logics (often intuitionistic in nature).
- **Our proposal**: define a version of PtS based on proofs in **sequent calculus** and investigate its properties.

Which version of PtS?

- **Base-extension Semantics**
  - Atomic bases (or bases);
  - Extensions;
  - Semantic clauses;

Base-extension Semantics

- A *atomic rule* is a natural deduction rule which concludes an atom using other atoms as premises.

$$\frac{p}{q} \qquad \frac{r \qquad s}{t} \qquad \frac{p \qquad \begin{array}{c}[q]\\ \vdots \\ r\end{array}}{s} \qquad \frac{p \qquad \begin{array}{c}\dfrac{q}{r}\,1\\ \vdots \\ s\end{array}}{t}\,1$$

- A atomic base S is a set of atomic rules.
- $\Gamma \vdash_S p$ holds if and only if p is derivable from the set of atoms Γ using the rules of S.
- A atomic base S' is an extension of a atomic base S if and only if $S' \supseteq S$.

Base-extension Semantics

Consider a set S with the following rules:

$$\frac{p}{q}\ \text{Rule 1} \qquad \frac{q \qquad r}{s}\ \text{Rule 2} \qquad \frac{\begin{matrix}[r]\\ \vdots \\ s\end{matrix}}{t}\ \text{Rule 3}$$

This is a derivation showing $p \vdash_S t$ in S:

$$\dfrac{\dfrac{p}{q}\ \text{Rule 1} \qquad [r]}{\dfrac{s}{t}\ \text{Rule 3}}\ \text{Rule 2}$$

- If S' contains only rules 1 and 2, then S is an extension of S'.

Philosophical motivations

- Atomic derivability is used to establish support in bases:

  $\vDash_S p$ iff $\vdash_S p$, for atomic p

- Semantic clauses expressing the proof conditions of connectives then extend support from atoms to formulas:

  $\vDash_S A \wedge B$ iff $\vDash_S A$ and $\vDash_S B$

- After base support is defined, logical validity is defined as support (that is, provability) in all bases:

  $\Gamma \vDash A$ iff $\Gamma \vDash_S A$ for all S.

Philosophical motivation

- The proof conditions of connectives were originally equated with the conditions expressed by their introduction rules, following a remark by Gentzen:

The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions. This fact may be expressed as follows: In eliminating a symbol, we may use the formula with whose terminal symbol we are dealing only 'in the sense afforded it by the introduction of that symbol'. (Gentzen, 1934/35)

Issues with introduction-based proof conditions

- **Definition of disjunction**

  (BeS definition) $\vDash_S (A \vee B)$ iff $\forall S'(S' \supseteq S)$: $A \vDash_{S'} p$ and $B \vDash_{S'} p$ implies $\vDash_{S'} p$, for all atomic p;

  If we define $\vDash_S A \vee B$ as holding iff either $\vDash_S A$ or $\vDash_S B$, intuitionistic logic is incomplete w.r.t. this semantics.

- **Definition of absurdity**

  (BeS definition) $\vDash_S \bot$ iff $\vDash_S p$, for all atomic p;

  If we define that $\nvDash_S \bot$ is the case for all bases, then $\vDash_S \neg\neg p$ holds for every atom p.

  We can, however, treat $\bot$ as an atom and require $\nvdash_S \bot$.

Semantic clauses

Definition (Sandqvist-style semantic clauses)

1. $\vDash_S p$ iff $\vdash_S p$, for atomic p;
2. $\vDash_S (A \wedge B)$ iff $\vDash_S A$ and $\vDash_S B$;
3. $\vDash_S (A \to B)$ iff $A \vDash_S B$;
4. $\vDash_S \bot$ iff $\vDash_S p$, for all atomic p;
5. $\vDash_S (A \vee B)$ iff $\forall S'(S' \supseteq S)$: $A \vDash_{S'} p$ and $B \vDash_{S'} p$ implies $\vDash_{S'} p$, for all atomic p;
6. $\Gamma \vDash_S A$ iff $\forall S'(S' \supseteq S)$: if $\vDash_{S'} B$ for all $B \in \Gamma$ then $\vDash_{S'} A$, for non-empty Γ;
7. $\Gamma \vDash A$ iff $\Gamma \vDash_S A$ for all S.

Harmony and semantic clauses

- Semantic definitions based on the introduction rules of natural deduction were expected to yield intuitionistic logic due to their harmonious character.
- In multiple succedent sequent calculus, on the other hand, classical logic is perfectly harmonious.
- As we will show, this harmony makes it possible to define classical semantic clauses for sequent BeS by relying only on introduction (right-introduction) rules.

Sequent derivability

- Before defining semantic clauses for sequent BeS, we must adapt atomic derivability notions of traditional BeS to the sequent setting.
- This is not as straightforward as it sounds!
- Problems: context handling and structural rules.

Context handling

- Consider the following two atomic rules:

$$\frac{t}{r} \qquad\qquad \frac{p, q \Rightarrow t}{\Rightarrow r}$$

- Due to how deducibility is defined in natural deduction, the left rule can be applied whenever we have a deduction of t depending on a set of atoms Γ to obtain a deduction with conclusion r that depends on Γ.
- Due to how deducibility is defined in sequent calculus, the right rule can only be applied if we have a deduction of the exact sequent $p, q \Rightarrow t$.
- In other words, context are arbitrary in natural deduction but not necessarily so in sequent calculus.

Atomic cuts and structural rules

- Consider a base S containing only the following two rules:

$$\frac{}{\Rightarrow p} \qquad \frac{}{p \Rightarrow q}$$

- Now consider the extension S' obtained by adding to S the following rule for every sets of atoms Γ, Γ', Δ and Δ'

$$\frac{\Gamma \Rightarrow p, \Delta \qquad \Gamma', p \Rightarrow \Delta'}{\Gamma, \Gamma' \Rightarrow \Delta, \Delta'}\ \textit{Atomic cut}$$

- We can show that $\nvdash_S \emptyset \Rightarrow q$ but $\vdash_{S'} \emptyset \Rightarrow q$, meaning that presence of the cut rule (and also of atomic structural rules) changes the semantics of bases!

Two dilemmas for sequent deductions

- Context handling is easier in the presence of atomic cut rules. However, atomic cut rules change the semantic content of bases, and since the logical calculus is supposed to be cut-free this inclusion is not ideal.
- Context handling is easier in the presence of rules with arbitrary contexts. However, such rules are not ideal if we are aiming to provide semantics for substructural logics.
- Since we are dealing with classical logic (for now), in this presentation we will consider both cut-free and cut-inclusive version of atomic bases **whose rules contain arbitrary contexts**.

Bases and deducibility

Definition (Base)

An *atomic sequent system* (or *sequent base*) $\mathcal{B}$ is a (possibly empty) set of atomic sequent rules of the form

$$\frac{\Gamma^1_{\mathsf{At}} \Rightarrow \Delta^1_{\mathsf{At}} \qquad \ldots \qquad \Gamma^n_{\mathsf{At}} \Rightarrow \Delta^n_{\mathsf{At}}}{\Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}}$$

Which is closed under rules of the following form for every atom p and every set Γ_{At} and Δ_{At} of atoms:

$$\frac{}{\Gamma_{\mathsf{At}}, p \Rightarrow p, \Delta_{\mathsf{At}}}\ \mathsf{Ainit}$$

Cut-free and cut-inclusive bases

Definition

A base is *cut-inclusive* if it contains the following rule for every set of atoms $\Gamma^1_{\mathsf{At}}, \Gamma^2_{\mathsf{At}}$, $\Delta^1_{\mathsf{At}}, \Delta^2_{\mathsf{At}}$ and all atoms p:

$$\frac{\Gamma^1_{\mathsf{At}} \Rightarrow \Delta^1_{\mathsf{At}}, p \qquad p, \Gamma^2_{\mathsf{At}} \Rightarrow \Delta^2_{\mathsf{At}}}{\Gamma^1_{\mathsf{At}}, \Gamma^2_{\mathsf{At}} \Rightarrow \Delta^1_{\mathsf{At}}, \Delta^2_{\mathsf{At}}} \text{Acut}$$

If a base is not *cut-inclusive* then it is *cut-free*.

Derivability (with arbitrary contexts)

Definition (Derivability)

For every $\mathcal{B}$, the relations $\vdash_{\mathcal{B}}$ is defined as follows.

1. For rule with empty premises in $\mathcal{S}$ of the form

$$\frac{}{\Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}}$$

$\vdash_{\mathcal{S}} \Theta_{\mathsf{At}}, \Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}, \Sigma_{\mathsf{At}}$ holds for any sets Θ_{At} and Σ_{At};

2. If $\vdash_{\mathcal{B}} \Theta^1_{\mathsf{At}}, \Gamma^1_{\mathsf{At}} \Rightarrow \Delta^1_{\mathsf{At}}, \Sigma^1_{\mathsf{At}}, \ldots, \vdash_{\mathcal{B}} \Theta^n_{\mathsf{At}}, \Gamma^n_{\mathsf{At}} \Rightarrow \Delta^n_{\mathsf{At}}, \Sigma^n_{\mathsf{At}}$ hold in $\mathcal{B}$ and the following rule is in $\mathcal{B}$:

$$\frac{\Gamma^1_{\mathsf{At}} \Rightarrow \Delta^1_{\mathsf{At}} \qquad \ldots \qquad \Gamma^n_{\mathsf{At}} \Rightarrow \Delta^n_{\mathsf{At}}}{\Gamma \Rightarrow \Delta}$$

then $\vdash_{\mathcal{B}} \Theta^1_{\mathsf{At}}, \ldots, \Theta^n_{\mathsf{At}}, \Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}, \Sigma^1_{\mathsf{At}}, \ldots, \Sigma^n_{\mathsf{At}}$ holds.

Example

Consider a base $\mathcal{B}$ with the following two rules:

$$\frac{p \Rightarrow q}{\Rightarrow s}\, R1 \qquad\qquad \frac{}{p, r \Rightarrow q}\, R2$$

The following is a deduction showing $\vdash_{\mathcal{B}} r \Rightarrow q$:

$$\dfrac{\dfrac{}{p, r \Rightarrow q}\, R2}{r \Rightarrow s}\, R1$$

And a single application of $R2$ shows $\vdash_{\mathcal{B}} \Gamma_{\mathsf{At}}, p, r \Rightarrow q, \Delta_{\mathsf{At}}$ for any Γ_{At} and any Δ_{At}.

Semantic clauses

Definition (Support)

Let $X_{\mathsf{At}} \subseteq \mathsf{At}$ for any set X. *Support* in a base $\mathcal{B}$ (written $\Vdash_{\mathcal{B}}$) is defined as follows:

1. $\Vdash_{\mathcal{B}} \Gamma_{\mathsf{At}}$ iff $\vdash_{\mathcal{B}} \emptyset \Rightarrow \Gamma_{\mathsf{At}}$;
2. $\Vdash_{\mathcal{B}} A \wedge B, \Gamma$ iff $\Vdash_{\mathcal{B}} A, \Gamma$ and $\Vdash_{\mathcal{B}} B, \Gamma$;
3. $\Vdash_{\mathcal{B}} A \vee B, \Gamma$ iff $\Vdash_{\mathcal{B}} A, B, \Gamma$;
4. $\Vdash_{\mathcal{B}} A \to B, \Gamma$ iff $A \Vdash_{\mathcal{B}} B, \Gamma$;
5. $\Vdash_{\mathcal{B}} \bot, \Gamma$ iff $\Vdash_{\mathcal{B}} \Gamma$
6. $\{A^1, \ldots, A^n\} \Vdash_{\mathcal{B}} \Delta$ iff for all $\mathcal{C} \supseteq \mathcal{B}$ and for all $\{\Theta^i_{\mathsf{At}}\}_{1 \leq i \leq n}$, if $\Vdash_{\mathcal{C}} \Theta^i_{\mathsf{At}}, A^i$ for all $1 \leq i \leq n$ then $\Vdash_{\mathcal{C}} \Theta^1_{\mathsf{At}}, \ldots, \Theta^n_{\mathsf{At}}, \Delta$;
7. $\Gamma \Vdash \Delta$ iff $\Gamma \Vdash_{\mathcal{B}} \Delta$ for all $\mathcal{B}$.

Comments

- The clauses for disjunction and $\perp$ no longer require quantification over atoms, and both are inspired by the system's introduction rules.
- Classical logic is sound and complete with respect to this semantics.

Completeness proof

- As mentioned before, the presence or absence of atomic cut in bases change their semantics.
- In fact, Sandqvist's traditional completeness proof can be straightforwardly adapted to cut-inclusive bases, but **not** to cut-free bases.
- It is still possible, however, to adapt the strategy so that completeness may be proved even with cut-free bases.

Cut-inclusive bases

- Sandqvist's strategy (adapted to sequents):

  Given a valid consequence $\Gamma \Vdash \Delta$, fix a mapping α from the subformulas of $\Gamma \cup \Delta$ to the set of atoms At such that $\alpha(p) = p$ for all atomic subformulas of $\Gamma \cup \Delta$ and $A \neq B$ implies $\alpha(A) \neq \alpha(B)$.

  Then, create a "simulation base"$\mathcal{U}$ in which the atom to which a formula A is mapped (denoted by p^A) behaves exactly as A does in the sequent calculus.

  Example:

$$\frac{\Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}, p^A \qquad \Gamma'_{\mathsf{At}} \Rightarrow \Delta'_{\mathsf{At}}, p^B}{\Gamma_{\mathsf{At}}, \Gamma'_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}, \Delta'_{\mathsf{At}}, p^{A \wedge B}}$$

Cut-inclusive bases

We prove the following lemma to establish that formulas indeed behave exactly like the atoms they are mapped to in simulation bases and their extensions:

Lemma

Let Θ_{At} be any (possibly empty) set of atoms, Σ any (non-empty) set containing only subformulas of $\Gamma \cup \Delta$ and $\Sigma_{\mathsf{At}} = \{p^B | B \in \Sigma\}$. Let $\mathcal{U}$ be a **cut-inclusive** *simulation base. Then, for all $\mathcal{B} \supseteq \mathcal{U}$, $\Vdash_{\mathcal{B}} \Sigma, \Theta_{\mathsf{At}}$ if and only if $\vdash_{\mathcal{B}} \emptyset \Rightarrow \Sigma_{\mathsf{At}}, \Theta_{\mathsf{At}}$.*

By using this lemma we can actually show that if $\Gamma \Vdash \Delta$ then $\vdash_{\mathcal{U}} \Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}$ is a derivable sequent in $\mathcal{U}$ (where Γ_{At} and Δ_{At} contain the atoms representing the formulas in Γ and Δ), and since $\mathcal{U}$ mimicks the rule of classical sequent calculus this yields a proof showing $\Gamma \Rightarrow \Delta$ in classical sequent calculus.

Cut-free bases

The main lemma used in Sandqvist's strategy makes essential use of atomic instances of cut. The following can be shown for cut-free bases:

Lemma

Let Θ_{At} be any (possibly empty) set of atoms, Σ any (non-empty) set containing only subformulas of $\Gamma \cup \Delta$ and $\Sigma_{\mathsf{At}} = \{p^B | B \in \Sigma\}$. Let $\mathcal{U}$ be a **cut-free** *simulation base. Then* **it is not the case that**, *for all $\mathcal{B} \supseteq \mathcal{U}$, $\Vdash_{\mathcal{B}} \Sigma, \Theta_{\mathsf{At}}$ if and only if $\vdash_{\mathcal{B}} \emptyset \Rightarrow \Sigma_{\mathsf{At}}, \Theta_{\mathsf{At}}$.*

The proof needs to be adapted if we want a cut of completeness that does not rely on atomic cuts.

Cut-free bases

- In order to prove completess without atomic cut, we use the same mapping α but now build a *quasi-simulation base* $\mathcal{Q}$, which instead of including left-introduction simulation rules include rules that look like a natural deduction elimination rule:

$$\frac{\Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}, p^{A \wedge B}}{\Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}, p^{A}}$$

Cut-free bases

- Quasi-simualation bases allows us to prove Sandqvist's fundamental lemma even without instances of cut.
- We start with a consequence $\Gamma \Vdash \Delta$ and extract a proof of $\vdash_{\mathcal{Q}} \Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}$.
- Since there are rules in $\mathcal{Q}$ that do not correspond to any classical sequent calculus rule, this deduction does not immediately yield a classical proof of the sequent $\Gamma \Rightarrow \Delta$.
- However, it yields a proof which *can be transformed* into a classical proof of the sequent if we substitute every "quasi" rule by an application of *Cut*!

Cut-free completeness

$$\cfrac{\begin{matrix}\vdots\\ \Theta \Rightarrow A \wedge B, \Sigma\end{matrix}}{\Theta \Rightarrow A, \Sigma}\; Q\wedge_1^*$$

Is transformed into

$$\cfrac{\begin{matrix}\vdots\\ \Theta \Rightarrow A \wedge B, \Sigma\end{matrix} \qquad \cfrac{\cfrac{}{A, B \Rightarrow A}\; init}{A \wedge B \Rightarrow A}\; \wedge L}{\Theta \Rightarrow A, \Sigma}\; Cut$$

Cut-free completeness

- In short, the cut-free completeness proof replaces applications of atomic cuts in the fundamental lemma by applications of (traditional) cut at the last step of the proof.
- Cut applications are "pushed upwards" from atomic bases to the logical calculus, so all semantically robust instances of atomic cut are now replaced by semantic redundant applications of the usual cut rule (which is admissible).

Future work: substructural logics

- Our definition of atomic derivability is not-context sensitive, which is acceptable in classical logic but not in substructural logics.
- In order to define substructural semantics, we could define semantic clauses which are indexed by the antecedents of sequents:

  $\Vdash_{\mathcal{B}}^{\Gamma_{\mathsf{At}}} \Delta_{\mathsf{At}}$ iff $\vdash_{\mathcal{B}} \Gamma_{\mathsf{At}} \Rightarrow \Delta_{\mathsf{At}}$.
- Just as the cut rule plays an importat role in classical semantics, *all structural rules* are expected to play a semantic role in this semantics (and some of their atomic versions might also need to be included in bases).

The end

Thanks!

References

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