A Gödel modal logic over witnessed crisp models

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Fuzzy modal logics

In approximate reasoning it is usual *to deal simultaneously with both fuzziness of propositions and modalities*, e.g., to assign a degree of truth to propositions like

"John is possibly tall" or *"John is necessarily tall"*

where *"John is tall"* is a *fuzzy proposition* or address features like *certainty*, *belief* or *similarity*, which have natural interpretations in terms of modalities.

A natural semantics for fuzzy modal operators

Combine the *Kripke semantics* for modal operators and one of the possible *algebraic semantics* for many-valued logics.

A preeminent choice for algebraic semantics is *Gödel algebra*, interpreting $x \wedge y$ with the continuous t-norm $\min\{x, y\}$ and $\rightarrow$ as its residuum, since...

... *this is the only fuzzy logic whose modal analogue admits the normality axiom* $\Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$ [Buo et al., 2011].

Gödel-Kripke semantics

Language: $\mathcal{V} = \{p_1, p_2, \dots\}$, $\wedge$, $\vee$, $\rightarrow$ $(\neg\varphi = \varphi \rightarrow \bot)$, $\Box$, $\Diamond$.

Gödel-Kripke model (GK-model): $\mathfrak{M} = \langle W, R, e\rangle$ where:

$W \neq \emptyset$	$R : W \times W \rightarrow [0,1]$	$e : W \times \mathcal{V} \rightarrow [0,1]$
worlds	accessibility relation	evaluation

$e(w, \bot) = 0$

$e(w, \alpha\star\beta) = e(w,\alpha) \star e(w,\beta)$, for $\star \in \{\wedge, \vee, \rightarrow, \neg\}$

$$a \wedge b = \min(a,b) \quad a \vee b = \max(a,b) \quad a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases} \quad \neg a = \begin{cases} 1 & \text{if } a = 0 \\ 0 & \text{if } a > 0 \end{cases}$$

$$e(w, \Box\alpha) = \inf_{x \in W}\Big\{ R(w,x) \rightarrow e(x,\alpha) \Big\} \qquad e(w, \Diamond\alpha) = \sup_{x \in W}\Big\{ R(w,x) \wedge e(x,\alpha) \Big\}$$

φ **is valid in** $\mathcal{M} = \langle W, R, e\rangle$ iff $\forall w \in W,\ e(w,\varphi) = 1 \qquad (\mathcal{M} \models \varphi)$

Gödel modal logics

Gödel Modal Logic - no restriction on R

$$\mathbf{GK} = \{\ \varphi \mid \varphi \text{ valid in all Gödel-Kripke models } \}$$

Crisp Gödel Modal Logic - R crisp

$$R : W \times W \to \{0, 1\} \qquad (R \subseteq W \times W)$$

$$e(w, \Box\alpha) = \inf\left(\{\ e(x, \alpha) \mid wRx\ \} \cup \{1\}\right) \qquad (= 1 \text{ if } R(w) = \emptyset)$$

$$e(w, \Diamond\alpha) = \sup\left(\{\ e(x, \alpha) \mid wRx\ \} \cup \{0\}\right) \qquad (= 0 \text{ if } R(w) = \emptyset)$$

$$\mathbf{G}^{c} = \{\ \varphi \mid \varphi \text{ valid in all } \underbrace{\textbf{crisp} \text{ Gödel-Kripke models}}_{\mathbf{G}^{c}\text{-models}}\ \}$$

GK and $\mathbf{G}^c$

Gödel Modal Logic GK - no restriction

[1] Axiomatizations for $\mathbf{GK}_\Box$ and $\mathbf{GK}_\Diamond$, finite model property (FMP) for $\mathbf{GK}_\Diamond$, no FMP for $\mathbf{GK}_\Box$

[2] Analytic calculi for $\mathbf{GK}_\Box$ and $\mathbf{GK}_\Diamond$, decidability, PSPACE-completeness

[3] Axiomatization for **GK**

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[1] Caicedo, Rodríguez: Standard Gödel modal logics. Studia Logica (2010)

[2] Metcalfe, Olivetti: Towards a proof theory of Gödel modal logics. LMCS (2011)

[3] Caicedo, Rodríguez: Bi-modal Gödel logic over [0,1]-valued Kripke frames. JLC (2015)

Crisp Gödel Modal Logic $\mathbf{G}^c$ - R crisp

[1] Axiomatization of $\mathbf{G}^c$, $\mathbf{GK} \subsetneq \mathbf{G}^c$, no FMP

[2] Analytic calculi for $\mathbf{G}^c_\Box$ and $\mathbf{G}^c_\Diamond$, decidability, PSPACE-completeness

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[1] Rodríguez, Vidal: Axiomatization of crisp Gödel modal logic. Studia Logica (2021)

[2] Metcalfe, Olivetti: Towards a proof theory of Gödel modal logics. LMCS (2011)

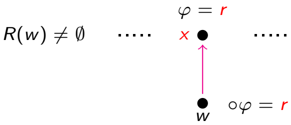
Our proposal

Witnessed crisp Gödel Modal Logic - R crisp and witnessed

$$R : W \times W \to \{0,1\} \qquad (R \subseteq W \times W)$$

$$\forall w : R(w) \neq \emptyset \qquad e(w, \circ\varphi) = r \;\Rightarrow\; \exists x : wRx \text{ and } e(x,\varphi) = r \qquad \circ \in \{\Box, \Diamond\}$$

$$e(w, \Box\alpha) = \min(\{\, e(x,\alpha) \mid wRx \,\} \cup \{1\})$$
$$e(w, \Diamond\alpha) = \max(\{\, e(x,\alpha) \mid wRx \,\} \cup \{0\})$$



$$\mathbf{GW}^{\mathrm{c}} = \{\, \varphi \mid \varphi \text{ valid in all } \underbrace{\textbf{witnessed crisp} \text{ Gödel-Kripke models}}_{\mathbf{GW}^{\mathrm{c}}\text{-models}} \,\}$$

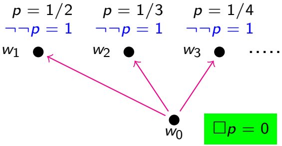
Remarks on $\mathbf{GW}^{\mathrm{c}}$-models (1)

Witnessed crisp Gödel Modal Logic - R crisp and witnessed

$\forall w : R(w) \neq \emptyset \quad e(w, \circ\varphi) = r \quad \Rightarrow \quad \exists w' : wRw' \text{ and } e(w', \varphi) = r \qquad \circ \in \{\Box, \Diamond\}$

Example of a NON witnessed $\mathbf{G}^{\mathrm{c}}$-model

$W = \{ w_j \mid j \geq 0 \} \qquad R = \{ (w_0, w_k) \mid k \geq 1 \} \qquad e(w_k, p) = \frac{1}{k+1} \quad k \geq 1$



Not witnessed: $e(w_0, \Box p) = \inf\{\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots\} = 0$ but $\forall k \geq 1,\ e(w_k, p) > 0$

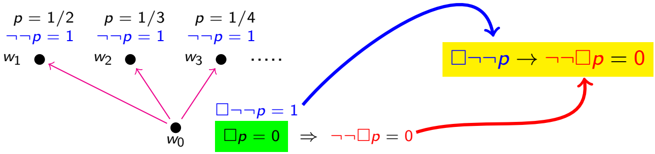
Remarks on $\mathbf{GW}^{c}$-models (2)

Witnessed crisp Gödel Modal Logic - R crisp and witnessed

$\forall w : R(w) \neq \emptyset \quad \boxed{e(w, \circ\varphi) = r} \quad \Rightarrow \quad \boxed{\exists w' : wRw' \text{ and } e(w', \varphi) = r} \quad \circ \in \{\Box, \Diamond\}$

$\mathbf{G}^{c}$-countermodel for $\varphi = \Box\neg\neg p \to \neg\neg\Box p$

$W = \{\, w_j \mid j \geq 0 \,\} \qquad R = \{\, (w_0, w_k) \mid k \geq 1 \} \qquad e(w_k, p) = \frac{1}{k+1} \quad k \geq 1$



$$\left.\begin{array}{r} \forall k\, e(w_k, p) > 0 \Rightarrow \forall k\, e(w_k, \neg\neg p) = 1 \Rightarrow e(w_0, \Box\neg\neg p) = 1 \\ e(w_0, \Box p) = 0 \Rightarrow e(w_0, \neg\neg\Box p) = 0 \end{array}\right\} \Rightarrow e(w_0, \varphi) = 0$$

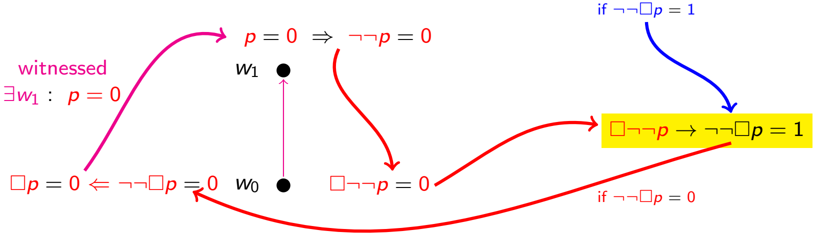
Accordingly $\varphi \notin \mathbf{G}^{c}$. Note that the above model is infinite.

Remarks on $\mathbf{GW}^c$-models (3)

Witnessed crisp Gödel Modal Logic - *R* crisp and witnessed

$\forall w : R(w) \neq \emptyset \quad e(w, \circ\varphi) = r \;\Rightarrow\; \exists w' : wRw' \text{ and } e(w', \varphi) = r \quad \circ \in \{\Box, \Diamond\}$

$\varphi = \Box\neg\neg p \to \neg\neg\Box p \in \mathbf{GW}^c$



Since $\varphi \in \mathbf{GW}^c \setminus \mathbf{G}^c$ and every $\mathbf{GW}^c$-model is a $\mathbf{G}^c$-model $\Rightarrow \mathbf{GW}^c \supsetneq \mathbf{G}^c \supsetneq \mathbf{GK}$

Since, every *finite crisp model* is *witnessed*, every $\varphi \in \mathbf{GW}^c \setminus \mathbf{G}^c$ has an infinite $\mathbf{G}^c$-countermodel.

The calculus $\mathcal{C}_{\mathrm{GW^c}}$

$\mathcal{C}_{\mathrm{GW^c}}$ is inspired to the calculus $\mathcal{T}(\mathbf{KG}^2_{\mathrm{fb}})$ for $\mathbf{KG}^2_{\mathrm{fb}}$ presented in [Bílková et al.,2022].

$\mathbf{KG}^2_{\mathrm{fb}}$ is an extension of $\mathbf{GW}^{\mathrm{c}}$ over a more expressive language, including an involutive negation and a co-implication.

$\mathcal{C}_{\mathrm{GW^c}}$ is refutation calculus acting on constraints over labelled formulas (labels representing worlds of Gödel-Kripke models).

Results overview

- termination
- completeness
- countermodel-construction and finite model property
- proof-search procedure (no-backtracking)
- PSPACE-decidability
- JTabWb implementation

The calculus $\mathcal{C}_{\mathrm{GW}^{\mathrm{c}}}$: constraints

$\blacktriangledown \in \{<, \leq, >, \geq\}$

constraint $u \blacktriangledown t$

atomic constraint $t \blacktriangledown t$

(c-term)
c constant in $[0,1]$, $r \in [0,1]$
$\underbrace{w : \varphi}_{\text{labelled formula}}$ (φ any formula), w world label

(atomic c-term)
c constant in $[0,1]$, $r \in [0,1]$
$\underbrace{w : p,\ w : \bot}_{\text{labelled formulas}}$, w world label

Examples

Constraint	*Intuitive semantical reading*
$c_1 > 0$ (atomic)	The value of constant c_1 is > 0
$w_0 : p \geq w_0 : q$ (atomic)	The value of propositional var. p at at world w_0 is $\geq$ than the value of propositional var. q at w_0
$w_2 : \Box p \to \Box\Box p < 1$	The value of wff $\Box p \to \Box\Box p$ at world w_2 is < 1

Constraints semantics

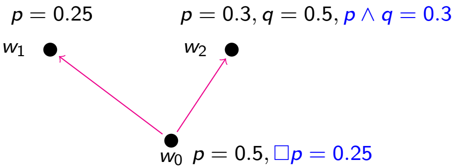
Given a set of constraints Γ and a $\mathbf{GW}^c$-model $\mathfrak{M} = \langle W, R, e \rangle$, $\mathcal{M}$ **satisfies** Γ ($\mathcal{M} \models \Gamma$) if there exists a mapping ι associating:

- a value in $[0, 1]$ to every rational constant in Γ
- a world label in Γ to W

such $\iota(u) \blacktriangledown \iota(t)$ for every constraint $u \blacktriangledown t \in \Gamma$, where $\iota(w : \alpha) = e(\iota(w), \alpha)$ (*all constraints are simultaneously satisfied in model* $\mathfrak{M}$).

Example

$$\Gamma = \{c_1 < 0.5\,,\ \boxed{w' : p > c_1}\,,\ \boxed{w' : \Box p} \geq 0.25\,,\ \boxed{w'' : p \wedge q} \leq 0.3\}$$



$$\begin{array}{ccc} & \iota & \\ c_1 & \mapsto & 0.4 \\ w' & \mapsto & w_0 \\ w'' & \mapsto & w_2 \end{array}$$

$$\begin{aligned} \iota(\Gamma) &= \{\iota(c_1) < 0.5\,,\ e(\iota(w'), p) > \iota(c_1)\,,\ e(\iota(w'), \Box p) \geq 0.25\,,\ e(\iota(w'') : p \wedge q) \leq 0.3\} \\ &= \{0.4 < 0.5\,,\ e(w_0, p) > 0.4\,,\ e(w_0, \Box p) \geq 0.25\,,\ e(w_2 : p \wedge q) \leq 0.3\} \\ &= \{0.4 < 0.5\,,\ 0.5 > 0.4\,,\ 0.25 \geq 0.25\,,\ 0.3 \leq 0.3\} \end{aligned}$$

The calculus $\mathcal{C}_{\mathrm{GW^c}}$

$\mathcal{C}_{\mathrm{GW^c}}$ is a *refutation calculus for constraint sets* Γ in the sense that:

Soundness

$$\vdash_{\mathrm{GW^c}} \Gamma \quad \Rightarrow \quad \text{there is no } \mathbf{GW}^{\mathrm{c}}\text{-model } \mathfrak{M} \text{ s.t. } \mathfrak{M} \models \Gamma$$

Application

If we can build a derivation for the constraint $\boxed{w : \varphi < 1}$

$\Downarrow$

there is no $\mathfrak{M}$ s.t. $\mathfrak{M} \models \boxed{w : \varphi < 1}$

$\Downarrow$

$\nexists\, (\mathfrak{M}\,,\; w)$ s.t. $e(w, \varphi) < 1$

$\Downarrow$

$\forall\, \mathfrak{M}\,,\; \forall\, w \;\; e(w, \varphi) = 1$

$\Downarrow$

$\varphi \in \mathbf{GW}^{\mathrm{c}}$

The calculus: axioms

$$\frac{}{\Gamma}\,\mathrm{Ax} \quad \text{if } \mathrm{At}^{+}(\Gamma) \text{ is not consistent}$$

$$\mathrm{At}^{+}(\Gamma) = \mathrm{At}(\Gamma) \bigcup \underbrace{\{1 > t \mid \boxed{w : \Box\alpha} > t \in \Gamma\} \cup \{0 < t \mid \boxed{w : \Diamond\alpha} < t \in \Gamma\}}_{\text{needed to guarantee the coherence}}$$

A set of atomic constraint Γ_{at} **is consistent** if we can define a function σ mapping:

$$c \mapsto q \in [0,1]_{\mathbb{Q}} \qquad \boxed{w : p} \mapsto q \in [0,1]_{\mathbb{Q}} \qquad w : \bot \mapsto 0$$

so that *all constraints in $\sigma(\Gamma_{\mathrm{at}})$ are simultaneously satisfied*.

The calculus: axioms (2)

A set of atomic constraint Γ_{at} **is consistent** if we can define a function σ mapping:

$$c \mapsto q \in [0,1]_{\mathbb{Q}} \qquad \boxed{w : p} \mapsto q \in [0,1]_{\mathbb{Q}} \qquad \boxed{w : \bot} \mapsto 0$$

so that *all constraints in $\sigma(\Gamma_{\mathrm{at}})$ are simultaneously satisfied*.

Examples

- $\Gamma = \{\, w_1 : p > c_0,\ w_2 : p \leq c_0,\ c_0 < 1,\ w_1 : \bot \leq 0 \,\}$ is consistent. Indeed, let:

  $$\sigma : c_0 \mapsto 0.5\,,\ w_1 : p \mapsto 0.7\,,\ w_2 : p \mapsto 0\,,\ w_1 : \bot \mapsto 0$$

  then $\sigma(\Gamma) = \{\, 0.7 > 0.5\,,\ 0 \leq 0.5\,,\ 0.5 < 1\,,\ 0 \leq 0\}$ is satisfied.
- $\Gamma = \{w : \Box\alpha > c, c \leq 1, c \geq 1\}$ is not consistent, Indeed $\mathrm{At}^+(\Gamma) = \{1 > c, c \leq 1, c \geq 1\}$ cannot be satisfied. Note that with $\sigma : c \mapsto 1$ $\sigma(\mathrm{At}(\Gamma))$ is satisfied, but there is no model satisfying $w : \Box\alpha > 1$ (and hence no model satisfying Γ).

Consistency of Γ_{at} can be checked by a Constraint Solver over $\mathbf{Q}$.

An atomic labelled formula $\boxed{w : p}$ can be considered as a constant name.

Rules for $\wedge, \vee, \rightarrow$-constraints

$\lhd \in \{<, \leq\}$ (lt relations) $\qquad$ $\rhd \in \{>, \geq\}$ (gt relations)

$$\frac{w : \alpha \lhd t,\ \Gamma \qquad w : \beta \lhd t,\ \Gamma}{w : \alpha \wedge \beta \lhd t,\ \Gamma}\ \wedge\lhd \qquad \frac{w : \alpha \rhd t,\ w : \beta \rhd t,\ \Gamma}{w : \alpha \wedge \beta \rhd t,\ \Gamma}\ \wedge\rhd$$

$$\frac{w : \alpha \lhd t,\ w : \beta \lhd t,\ \Gamma}{w : \alpha \vee \beta \lhd t,\ \Gamma}\ \vee\lhd \qquad \frac{w : \alpha \rhd t,\ \Gamma \qquad w : \beta \rhd t,\ \Gamma}{w : \alpha \vee \beta \rhd t,\ \Gamma}\ \vee\rhd$$

$$\frac{w : \beta \leq b,\ w : \alpha > b,\ b < t,\ \Gamma}{w : \alpha \to \beta < t,\ \Gamma}\ \to <\ (\dagger)$$

$$\frac{t \geq 1,\ \Gamma \qquad w : \beta \leq b,\ w : \alpha > b,\ b \leq t,\ \Gamma}{w : \alpha \to \beta \leq t,\ \Gamma}\ \to \leq (\dagger)$$

$$\frac{w : \alpha \leq a,\ w : \beta \geq a,\ 1 \rhd t,\ \Gamma \qquad w : \beta \rhd t,\ \Gamma}{w : \alpha \to \beta \rhd t,\ \Gamma}\ \to \rhd\ (\ddagger)$$

$$(\dagger)\ b = \begin{cases} w : \beta & \text{if } \beta \text{ atomic} \\ \text{new const.} & \text{otherwise} \end{cases} \qquad (\ddagger)\ a = \begin{cases} w : \alpha & \text{if } \alpha \text{ atomic} \\ \text{new const.} & \text{otherwise} \end{cases}$$

Semantical intuition: rule $\rightarrow\leq$

Lemma (Soundness of the rules)

If $\mathfrak{M} \models \Gamma$, where Γ is the conclusion of a rule ρ, then $\exists$ a premise Γ' of ρ s.t. $\mathfrak{M} \models \Gamma'$.

$$a \rightarrow b = \begin{cases} 1 & \text{if } a \leq b \\ b & \text{if } a > b \end{cases}$$

$w \bullet \underbrace{\alpha \leq \beta, t = 1}_{\alpha \rightarrow \beta = 1}, \dots$

$w \bullet \underbrace{\beta \leq b, \alpha > b, b \leq t}_{\exists b : \alpha \rightarrow \beta \leq b < 1}, \dots$

$\alpha \rightarrow \beta = 1$

$\alpha \rightarrow \beta < 1$

$$\frac{t \geq 1, \dots \qquad w : \beta \leq b,\ w : \alpha > b,\ b \leq t, \dots}{w : \alpha \rightarrow \beta \leq t, \dots} \rightarrow \leq (\dagger)$$

The calculus: rules for $\Box$ and $\Diamond$

$$\frac{1 \lhd t,\ \Phi^{0,1}(\Gamma) \qquad w_1 : \alpha \lhd t,\ \Phi^{\Box,\Diamond}(\Gamma, w, w_1),\ \Gamma}{w : \Box\alpha \lhd t,\ \Gamma}\ \Box\lhd \qquad \lhd \in \{<, \leq\}$$

$$\frac{0 \rhd t,\ \Phi^{0,1}(\Gamma) \qquad w_1 : \alpha \rhd t,\ \Phi^{\Box,\Diamond}(\Gamma, w, w_1),\ \Gamma}{w : \Diamond\alpha \rhd t,\ \Gamma}\ \Diamond\rhd \qquad \rhd \in \{>, \geq\}$$

w_1 is a *new label (reading the rule* $\uparrow$*)*

Idea: w_1 represents an R-successor of w - we say that w generates w_1

$$\Phi^{\Box,\Diamond}(\Gamma, w, w_1) \ = \ \{\, w_1 : \beta \rhd t \mid w : \Box\beta \rhd t \in \Gamma \,\} \cup \{\, w_1 : \beta \lhd t \mid w : \Diamond\beta \lhd t \in \Gamma \,\}$$

Idea: the R-successor w_1 must coherently treat any $w : \circ\gamma$

$$\Phi^{0,1}(\Gamma) \ = \ \text{in } \Gamma \text{ replace} \begin{cases} w' : \Diamond\alpha \blacktriangledown t \text{ with } 0 \blacktriangledown t \\ w' : \Box\alpha \blacktriangledown t \text{ with } 1 \blacktriangledown t \end{cases}$$

Idea: if $R(w) = \emptyset$, every $\boxed{w' : \circ\gamma}$ $\blacktriangledown t$ *must hold with* $\boxed{w' : \Diamond\gamma} = 0$ *and* $\boxed{w' : \Box\gamma} = 1$

Semantical intuition: rule $\Box\lhd$

$$\Box\alpha = \min\left(\{\, e(w',\alpha) \mid wRw' \,\} \cup \{1\}\right)$$
$$\Diamond\alpha = \max\left(\{\, e(w',\alpha) \mid wRw' \,\} \cup \{0\}\right)$$

$1 \leq t$, $1 > x$, $0 < y$

$\Box\alpha = 1$, $\Box\beta = 1$, $\Diamond\gamma = 0$ (since $R(w) = \emptyset$)

w • $\Box\alpha \leq t$, $\Box\beta > x$, $\Diamond\gamma < y$ (by hyp.)

w_1 • $\alpha \leq t$, $\beta > x$, $\gamma < y$

w • $\Box\alpha \leq t$, $\Box\beta > x$, $\Diamond\gamma < y$ (by hyp.)

w has no R-sucessors

w has R-sucessors

$$\dfrac{1 \leq t,\ \overbrace{1 > x,\ 0 < y}^{\Phi^{0,1}(\Gamma)} \qquad w_1 : \alpha \leq t,\ \overbrace{w_1 : \beta > x,\ w_1 : \gamma < y}^{\Phi^{\Box,\Diamond}(\Gamma, w, w_1)}, \Gamma}{w : \Box\alpha \leq t,\ \underbrace{w : \Box\beta > x,\ w : \Diamond\gamma < y}_{\Gamma}}\ \Box\lhd$$

$$\Phi^{\Box,\Diamond}(\Gamma, w, w') = \{\, w' : \beta \rhd t \mid w : \Box\beta \rhd t \in \Gamma \,\} \cup \{\, w' : \beta \lhd t \mid w : \Diamond\beta \lhd t \in \Gamma \,\}$$

$$\Phi^{0,1}(\Gamma) = \text{in } \Gamma \text{ replace } w' : \Diamond\alpha \blacktriangledown t \text{ with } 0 \blacktriangledown t \text{ and } w' : \Box\alpha \blacktriangledown t \text{ with } 1 \blacktriangledown t$$

$\mathcal{C}$ is strongly terminating

Theorem

There exists a well founded relation $\prec_{\mathrm{c}}$ s.t. for every application ρ of a rule of $\mathcal{C}_{\mathrm{GW^c}}$, if Γ is the conclusion of ρ and Γ' is any of its premises, then $\Gamma' \prec_{\mathrm{c}} \Gamma$.

As a consequence: **any backward proof search strategy for $\mathcal{C}_{\mathrm{GW^c}}$ terminates**. *No backtracking is required* and only one proof-tree can be generated.

The well-founded relation $\prec_{\mathrm{c}}$

Size of multiset of constraints

$||\Gamma|| = \{\, |\Gamma[w]| \mid w$ is a world label in $\Gamma\,\}$ (multiset)

$|\Gamma[w]|$ = number of logical connectives in $\Gamma[w]$ (wffs labelled with w in Γ)

Well-founded relation on multiset of natural numbers [Baader-Nipkow 1998]

$$\Theta_1 \prec_{\mathrm{m}} \Theta_2 \quad \text{iff} \quad \Theta_1 \neq \Theta_2 \ \wedge\ (\, \forall k_1 \in \Theta_1 \setminus \Theta_2.\ \exists k_2 \in \Theta_2 \setminus \Theta_1.\ k_1 < k_2\,)\,.$$

Well-founded relation on multiset of constraints

$$\Gamma_1 \prec_{\mathrm{c}} \Gamma_2 \quad \text{iff} \quad ||\Gamma_1|| \prec_{\mathrm{m}} ||\Gamma_2||.$$

Countermodel construction

Let Bs be a *backward proof search strategy* for $\mathcal{C}_{\mathrm{GW^c}}$ where *a modal rule is backward applied iff no propositional rule can be applied* (plain proof-search strategy).

A branch $\mathcal{B} = \langle \Gamma_0, \ldots, \Gamma_n \rangle$ of a proof-tree $\mathcal{T}$ generated by Bs is **reduced** if Γ_0 is the root of $\mathcal{T}$, no rule can be backward applied to Γ_n, and Γ_n is not an axiom.

From a reduced branch $\mathcal{B} = \langle \Gamma_0, \ldots, \Gamma_n \rangle$ *we can extract* a discrete (e values in Q) $\mathbf{GW}^{\mathrm{c}}$-model $\mathrm{Mod}(\mathcal{B})$ such that $\mathrm{Mod}(\mathcal{B}) \models \Gamma_0$.

Theorem (Completeness and finite model property)

- *If* $\nvdash_{\mathrm{GW^c}} \Gamma$, *then there exists a discrete model for* Γ.
- *If* $\nvdash_{\mathrm{GW^c}} w : \varphi < 1$, *then* $\varphi \notin \mathbf{GW}^{\mathrm{c}}$.
- *If* $\varphi \notin \mathbf{GW}^{\mathrm{c}}$, *then* $\exists\, \mathcal{M}, w$ *s.t.* $e(w, \varphi) < 1$ ($\mathcal{M}$ is a countermodel for φ).

Countermodel construction: example

$c_0 \overset{\sigma}{\mapsto} 0.5 \qquad \boxed{w_2 : p} \overset{\sigma}{\mapsto} 0.3 \qquad \boxed{w_1 : p} \overset{\sigma}{\mapsto} 0.7$

$$\mathcal{B}\begin{cases}
\dfrac{\dfrac{\dfrac{\overbrace{\boxed{w_2 : p} \leq c_0}^{0.3\leq 0.5},\ \overbrace{\boxed{w_1 : p} > c_0}^{0.7>0.5},\ w_0 : \Box p > c_0,\ \overbrace{c_0 < 1}^{0.5<1}}{\boxed{w_1 : \Box p \leq c_0},\ w_1 : p > c_0,\ w_0 : \Box p > c_0,\ c_0 < 1}\ \Box\lhd\ (w_1 \text{ generates } w_2)}{\boxed{w_0 : \Box\Box p \leq c_0},\ w_0 : \Box p > c_0,\ c_0 < 1}\ \Box\lhd\ (w_0 \text{ generates } w_1)}{\boxed{w_0 : \Box p \to \Box\Box p < 1}}\ \to<
\end{cases}$$

$$\begin{tikzcd}
w_2 \quad \bullet \quad p = 0.3 \\
w_1 \quad \bullet \quad p = 0.7 \arrow[u] \\
w_0 \quad \bullet \quad p = 0 \arrow[u]
\end{tikzcd}$$

$\boxed{\begin{array}{l} W = \text{world labels occurring in } \mathcal{B} \\ wRw' \text{ if } w' \text{ is generated by } w \\ e(w,p) = \sigma(w : p) \text{ if } w : p \in \mathrm{At}(\Delta),\ e(w,p) = 0 \text{ oth.} \end{array}}$

$e(w_0, \Box p) = e(w_1, p) = 0.7$

$e(w_0, \Box\Box p) = e(w_1, \Box p) = e(w_2, p) = 0.3$

$e(w_0, \Box p \to \Box\Box p) = e(w_0, \Box\Box p) = 0.3$

Complexity and implementation

Complexity

The countermodel $\mathfrak{M}$ for $w_0 : \varphi < 1$ has

- *depth* $\leq |\varphi|$
- every world of $\mathfrak{M}$ has at most $|\varphi|$ R-successors

this implies that the size of $\mathfrak{M}$ is $O(|\varphi|^{|\varphi|})$.

By adapting the procedure described in [Bílková et al.,2022] we can prove that the decision problem for $\mathbf{GW}^c$ is in PSPACE.

Implementation: **gwcref**

Implementation of our proof-search procedure in JTabWb [Ferrari et al., 2017].

- standard backward depth-first proof search and countermodel extraction
- consistency of atomic constraints is checked using the Choco-solver Java library
- LaTeXgeneration of proof-search trees and countermodels

An "*intuitionistic modal logic style*" semantics

Intuitionistic modal logics (IML)

IPL extended with modalities.

IML Kripke-style semantics

Bi-relational structures $\mathcal{K} = \langle X, \leq, S, V \rangle$ with two accessibility relations:

- *intuitionistic relation* $\leq$: a partial order on X
- *the modal relation* S: a binary relation on X
- $V : X \to 2^{\mathcal{V}}$ s.t. $x \leq y$ implies $V(x) \subseteq V(y)$ (persistence)

where $\leq$ and R meet some connections relation (in the style of Fisher-Servi formalization).

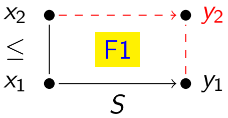
Motivation of an alternative semantics

Extend to the modal case the correspondence holding for Gödel multivalued logic

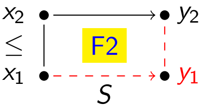
Semantics on Gödel T-norm	$\equiv$	Intuitionisitic semantics on linearly ordered Kripke models

to enable the use of IML methods for Gödel modal logics and their calculi, as in the non-modal case.

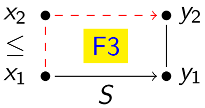
GW^c-bimodel: conditions on relations



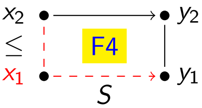
$x_1 \leq x_2 \wedge x_1 S y_1 \Rightarrow \exists y_2 : x_2 S y_2 \wedge y_1 \leq y_2$



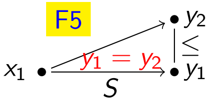
$x_1 \leq x_2 \wedge x_2 S y_2 \Rightarrow \exists y_1 : x_1 S y_1 \wedge y_1 \leq y_2$



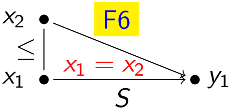
$x_1 S y_1 \wedge y_1 \leq y_2 \Rightarrow \exists x_2 : x_1 \leq x_2 \wedge x_2 S y_2$



$x_2 S y_2 \wedge y_1 \leq y_2 \Rightarrow \exists x_1 : x_1 \leq x_2 \wedge x_1 S y_1$



$x_1 S y_1 \wedge x_1 S y_2 \wedge y_1 \leq y_2 \Rightarrow y_1 = y_2$



$x_1 S y_1 \wedge x_2 S y_1 \wedge x_1 \leq x_2 \Rightarrow x_1 = x_2$

There are more conditions than in IML. *Are they all needed ?* Maybe not.

Equivalent semantics

The forcing relation $\Vdash$ between worlds of $\mathcal{K}$ and formulas is defined as follows:

$\mathcal{K}, x \nVdash \bot$

$\mathcal{K}, x \Vdash p$ iff $p \in V(x)$, where $p \in \mathcal{V}$

$\mathcal{K}, x \Vdash \alpha \wedge \beta$ iff $\mathcal{K}, x \Vdash \alpha$ and $\mathcal{K}, x \Vdash \beta$

$\mathcal{K}, x \Vdash \alpha \vee \beta$ iff $\mathcal{K}, x \Vdash \alpha$ or $\mathcal{K}, x \Vdash \beta$

$\mathcal{K}, x \Vdash \alpha \to \beta$ iff $\forall y \in X$ s.t. $y \geq x$, if $\mathcal{K}, y \Vdash \alpha$ then $\mathcal{K}, y \Vdash \beta$

$\mathcal{K}, x \Vdash \Box\alpha$ iff $\forall y \in X$, if xSy then $\mathcal{K}, y \Vdash \alpha$

$\mathcal{K}, x \Vdash \Diamond\alpha$ iff $\exists y \in X$ s.t. xSy and $\mathcal{K}, y \Vdash \alpha$

φ **is valid in** $\mathcal{K} = \langle X, \leq, S, V \rangle$ iff $\forall x \in X$, $\mathcal{K}, x \Vdash \varphi$ $\qquad (\mathcal{K} \Vdash \varphi)$

Theorem (Equivalence between $\mathbf{GW}^c$-models and bimodels)

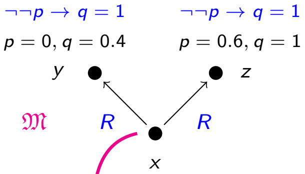
$$\begin{aligned}\mathbf{GW}^c &= \{\ \varphi \mid \varphi \text{ is valid in all witnessed crisp Gödel-Kripke models}\ \} \\ &= \{\ \varphi \mid \varphi \text{ is valid in all bimodels}\ \}\end{aligned}$$

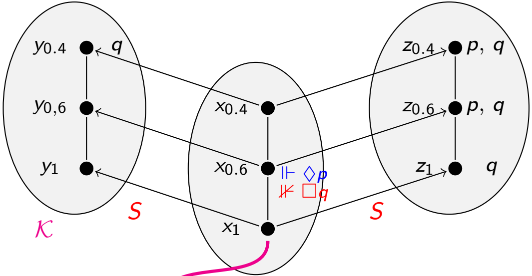
The proof is based on the construction of a correspondence between $\mathbf{GW}^c$-models and birelational models.

Example

W $(\{x, y, z\}) \Rightarrow$ clusters of $\mathcal{K}$ (a world $\forall$ non null value of e $(\{x_{0.4}, x_{0.6}, x_1\}, \dots)$

$w_a \leq w_b$ if $b \leq a$ $\quad$ $w_a S v_a$ if wRv $\quad$ $w_a \Vdash q$ if $e(w, q) \geq a$





$\Diamond p = 0.6,\ \Box q = 0.4$

$\Diamond p \to \Box q = 0.4$

$\Box(\neg\neg p \to q) = 1$

$\Box(\neg\neg p \to q) \to (\Diamond p \to \Box q) = 0.4$

$\forall x,\ x \Vdash \neg\neg p \to q$

$x_1 \nVdash \Diamond p \to \Box q$

$x_1 \Vdash \Box(\neg\neg p \to q)$

$x_1 \nVdash \Box(\neg\neg p \to q) \to (\Diamond p \to \Box q)$

$\Box(\neg\neg p \to q) \to (\Diamond p \to \Box q) \notin \mathbf{GW}^c$

Future work

Axiomatization for $\mathbf{GW}^{\mathrm{c}}$

We have axiomatizations for **GK** and $\mathbf{G}^{\mathrm{c}}$ from [Caicedo et al., 2010 and 2015] and [Metcalfe-Olivetti, 2011] but not for $\mathbf{GW}^{\mathrm{c}}$.

$$
\left.\begin{array}{l}
\left.\begin{array}{l}
\text{Intuitionistic propositional logic} \\
\text{(linearity axiom)} \quad (\alpha \to \beta) \vee (\beta \to \alpha)
\end{array}\right\} \mathbf{GL} \\
\left.\begin{array}{lll}
(K_\Box)\ \Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta) & (K_\Diamond)\ \Diamond(\alpha \vee \beta) \to (\Diamond\alpha \vee \Diamond\beta) & (F_\Diamond)\ \neg\Diamond\bot \\
(FS_1)\ \Diamond(\alpha \to \beta) \to (\Box\alpha \to \Diamond\beta) & (FS_2)\ (\Diamond\alpha \to \Box\beta) \to \Box(\alpha \to \beta) & \\
(N_\Box)\ \vdash \alpha \text{ implies } \vdash \Box\alpha & (N_\Diamond)\ \vdash \alpha \to \beta \text{ implies } \vdash \Diamond\alpha \to \Diamond\beta &
\end{array}\right\} \mathbf{GK} \\
\Box(\alpha \vee \beta) \to (\Box\alpha \vee \Diamond\beta)
\end{array}\right\} \mathbf{G}^{\mathrm{c}}
$$

Extension of the witness semantics (and of the calculus) to the fuzzy case

First order extension of $\mathbf{GW}^{\mathrm{c}}$

Refinements of the birelational semantics (IML-style calculi?)