Refined Tableau Systems for Some Modal Logics of Confluence

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- **Aim**: Systematic development of refined tableau systems for modal logics of confluence
- For many model logics of confluence, refined systems have had less attention
- **Our focus**: Developing refined tableau systems for this subset of the modal logics of confluence

Why refined tableau systems?

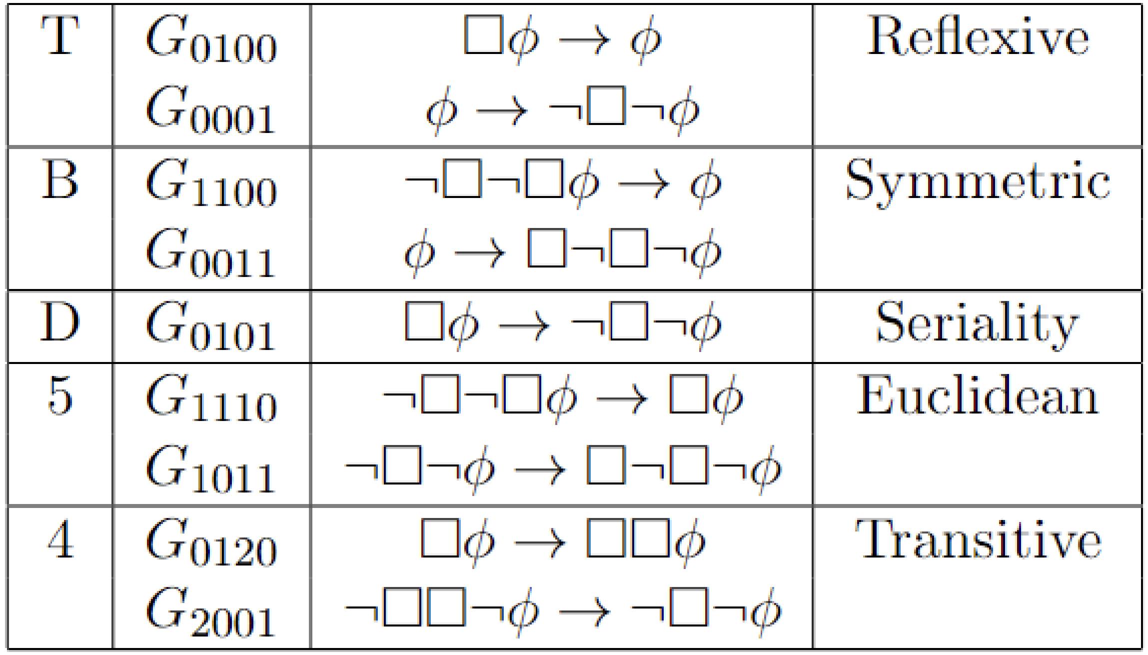
- Tableau is a popular method

- Refined tableau systems:
  - Perform fewer inferences to determine satisfiability
  - Construct smaller models: fewer worlds/labels, fewer relational links (or both)

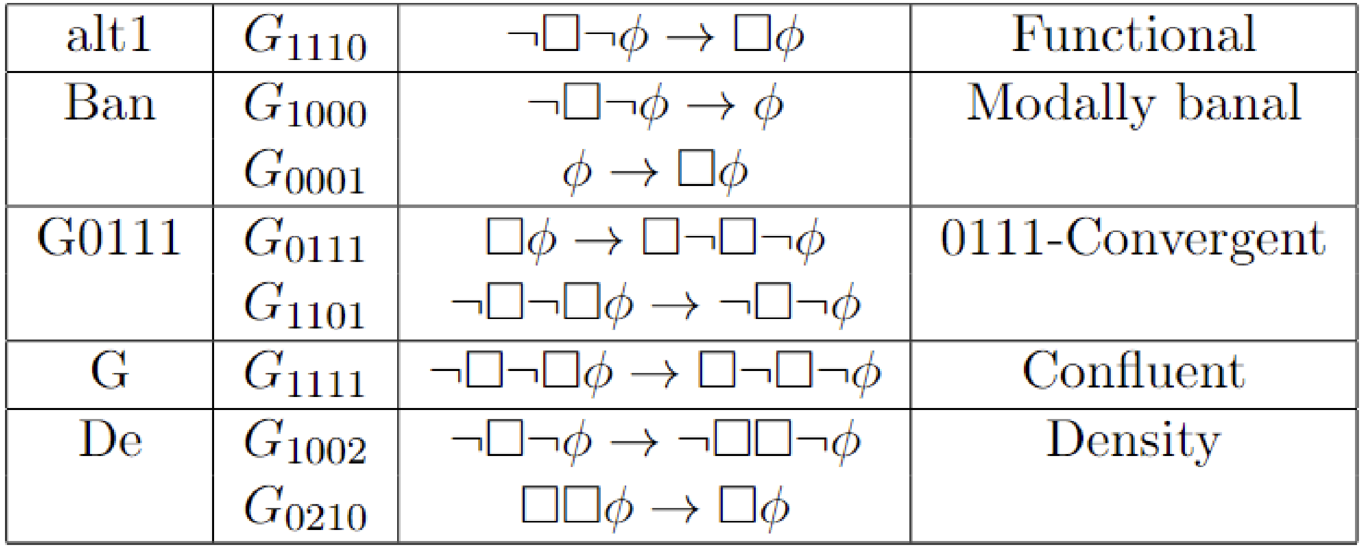
Modal logics of confluence

- Extensions of modal logic K with axioms of the form $\Diamond^p \Box^q \rightarrow \Box^r \Diamond^s$

  where p, q, r and s are natural numbers
- Encompasses large class of standard modal logics

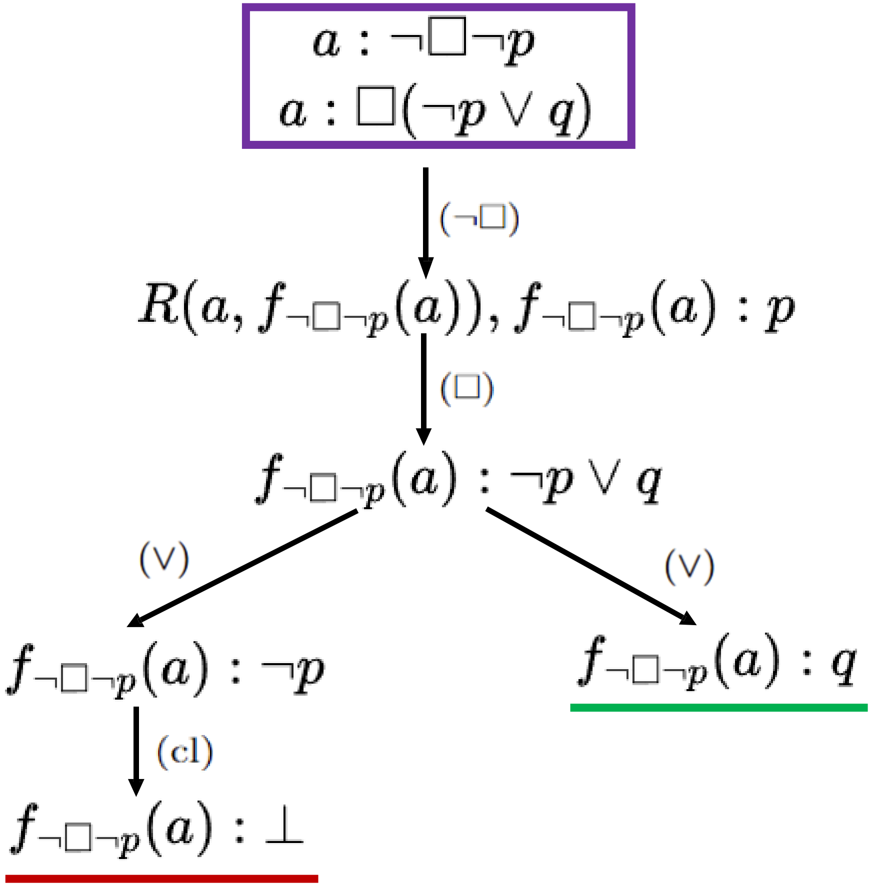


T	G_{0100} G_{0001}	$\Box\phi \to \phi$ $\phi \to \neg\Box\neg\phi$	Reflexive
B	G_{1100} G_{0011}	$\neg\Box\neg\Box\phi \to \phi$ $\phi \to \Box\neg\Box\neg\phi$	Symmetric
D	G_{0101}	$\Box\phi \to \neg\Box\neg\phi$	Seriality
5	G_{1110} G_{1011}	$\neg\Box\neg\Box\phi \to \Box\phi$ $\neg\Box\neg\phi \to \Box\neg\Box\neg\phi$	Euclidean
4	G_{0120} G_{2001}	$\Box\phi \to \Box\Box\phi$ $\neg\Box\Box\neg\phi \to \neg\Box\neg\phi$	Transitive



alt1	G_{1110}	$\neg\Box\neg\phi \to \Box\phi$	Functional
Ban	G_{1000} G_{0001}	$\neg\Box\neg\phi \to \phi$ $\phi \to \Box\phi$	Modally banal
G0111	G_{0111} G_{1101}	$\Box\phi \to \Box\neg\Box\neg\phi$ $\neg\Box\neg\Box\phi \to \neg\Box\neg\phi$	0111-Convergent
G	G_{1111}	$\neg\Box\neg\Box\phi \to \Box\neg\Box\neg\phi$	Confluent
De	G_{1002} G_{0210}	$\neg\Box\neg\phi \to \neg\Box\Box\neg\phi$ $\Box\Box\phi \to \Box\phi$	Density

Basic tableau system for modal logic K



$a : \neg\Box\neg p$

$a : \Box(\neg p \vee q)$

$(\neg\Box)$

$R(a, f_{\neg\Box\neg p}(a)), f_{\neg\Box\neg p}(a) : p$

$(\Box)$

$f_{\neg\Box\neg p}(a) : \neg p \vee q$

$(\vee)$

$(\vee)$

$f_{\neg\Box\neg p}(a) : \neg p$

$f_{\neg\Box\neg p}(a) : q$

(cl)

$f_{\neg\Box\neg p}(a) : \bot$

$$(\Box)\,\frac{s : \Box\phi, R(s,t)}{t : \phi} \qquad (\neg\Box)\,\frac{s : \neg\Box\phi}{R(s, f_{\neg\Box\phi}(s)), f_{\neg\Box\phi}(s) : \sim\phi}$$

$$(\vee)\,\frac{s : \phi \vee \psi}{s : \phi \;\mid\; s : \psi} \qquad (\mathrm{cl})\,\frac{s : \phi, s : \neg\phi}{\bot}$$

- Two main types of rules: structural and propagation

Tableau rules reflecting the correspondence properties (structural rules)

$$\forall x, y, z(R^i(x,y) \wedge R^k(x,z) \rightarrow \exists u(R^j(y,u) \wedge R^l(z,u))$$

T	$(\text{refl})\,\dfrac{}{R(s,s)}$
B	$(\text{symm})\,\dfrac{R(s,t)}{R(t,s)}$
D	$(\text{seri})\,\dfrac{}{R(s,f(s))}$
5	$(\text{eucl})\,\dfrac{R(s,t), R(s,u)}{R(t,u)}$
4	$(\text{trans})\,\dfrac{R(s,t), R(t,u)}{R(s,u)}$

alt1	$(\text{func})\,\dfrac{R(s,t), R(s,u)}{t \approx u}$
Ban	$(\text{mban})\,\dfrac{R(s,t)}{s \approx t}$
G0111	$(\text{conv})\,\dfrac{R(s,t)}{R(s,g(s,t)), R(t,g(s,t))}$
G	$(\text{conf})\,\dfrac{R(s,t), R(s,u)}{R(t,h(s,t,u)), R(u,h(s,t,u))}$
De	$(\text{dens})\,\dfrac{R(s,t)}{R(s,i(s,t)), R(i(s,t),t)}$

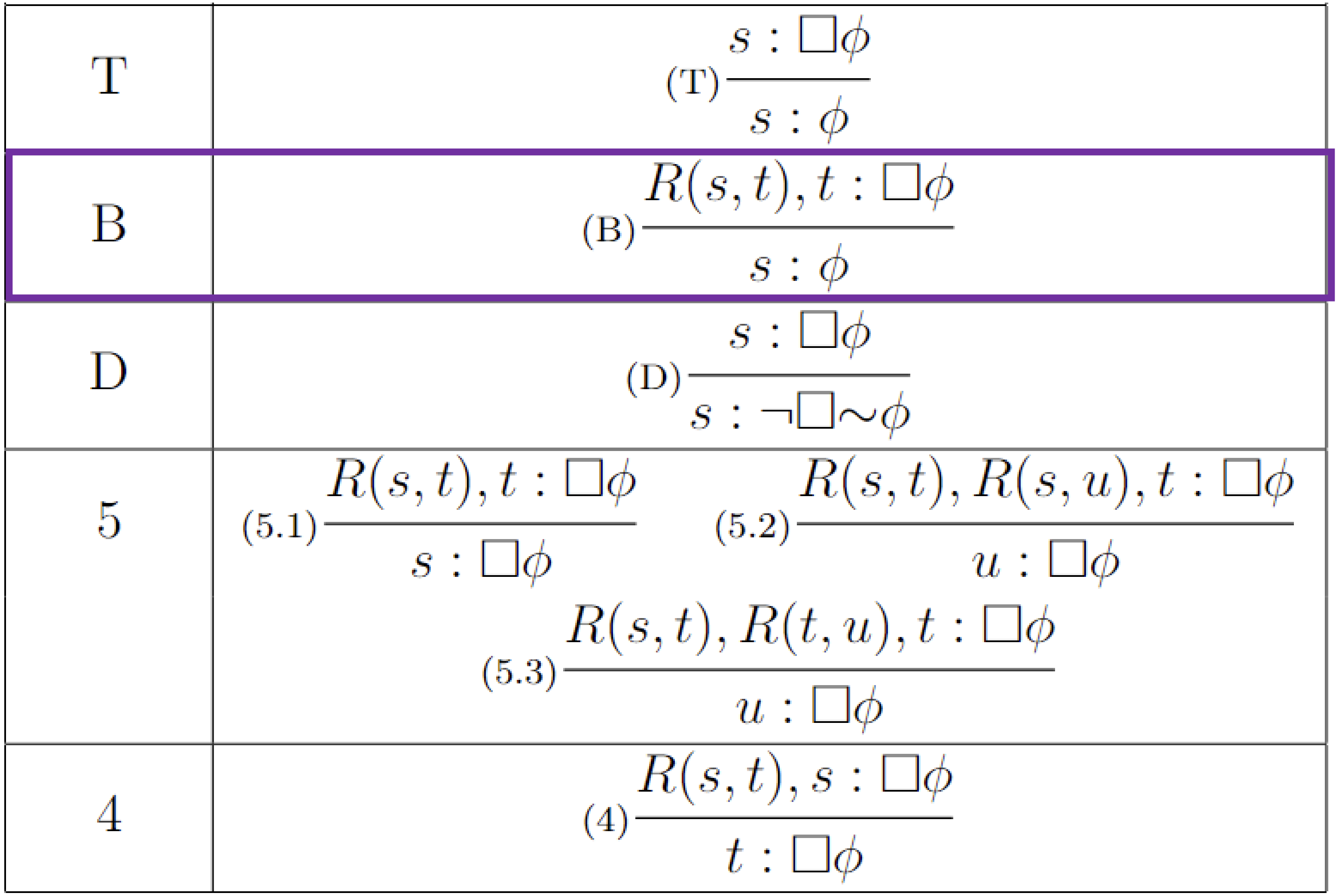
$$\text{T} : \forall x, y, z(R^0(x,y) \wedge R^0(x,z) \rightarrow \exists u(R^1(y,u) \wedge R^0(z,u))$$

$$\forall x R(x,x)$$

Classical tableau systems based on rules reflecting correspondence properties

- **Advantages:** Easy to develop for first-order definable logics, because the extra rules simply reflect the correspondence properties
- If formula satisfiable, model can be extracted from tableau derivation
- **Disadvantages**: When the correspondence properties are triangular e.g. transitivity, density, 0111-convergence, confluence…, the models tend to get very large and are easily infinite.
- For these cases, tableau provers may perform worse

Propagation tableau rules



T	$$(\mathrm{T})\ \frac{s : \Box\phi}{s : \phi}$$
B	$$(\mathrm{B})\ \frac{R(s,t), t : \Box\phi}{s : \phi}$$
D	$$(\mathrm{D})\ \frac{s : \Box\phi}{s : \neg\Box{\sim}\phi}$$
5	$$(5.1)\ \frac{R(s,t), t : \Box\phi}{s : \Box\phi} \qquad (5.2)\ \frac{R(s,t), R(s,u), t : \Box\phi}{u : \Box\phi}$$ $$(5.3)\ \frac{R(s,t), R(t,u), t : \Box\phi}{u : \Box\phi}$$
4	$$(4)\ \frac{R(s,t), s : \Box\phi}{t : \Box\phi}$$

- Combination of labelled formulae and relational links
- Are known to perform better than their structural counterparts

Classical system for KG0111

G0111

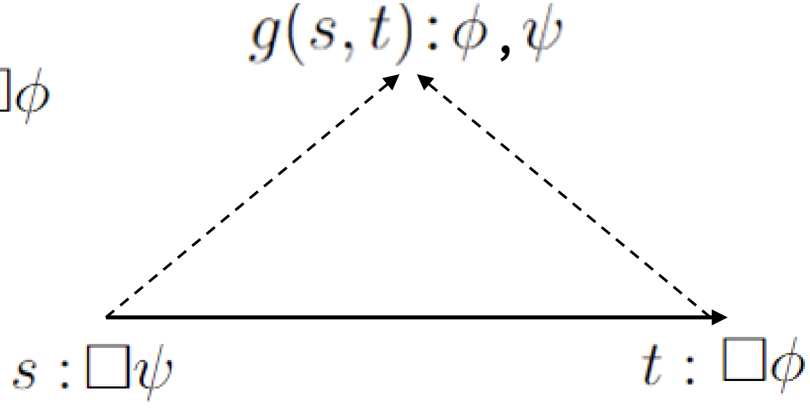
$$\Box\phi \to \Box\Diamond\phi$$

$$\Diamond\Box\phi \to \Diamond\phi$$

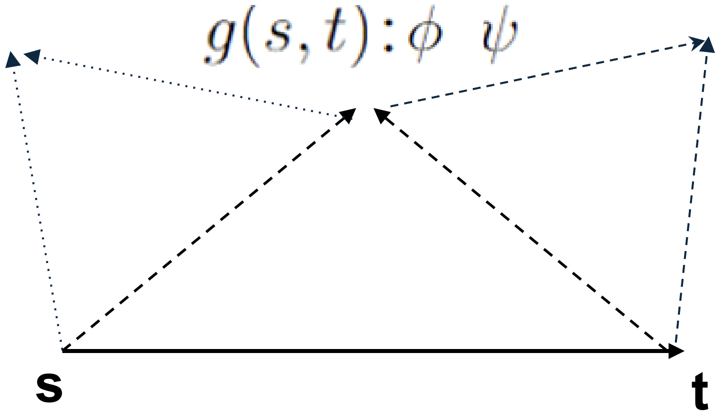
0,1,1,1-Convergent

$$\forall x, y(R(x,y) \to \exists z(R(x,z) \wedge R(y,z)))$$

$$\text{(conv)}\,\frac{R(s,t)}{R(s,g(s,t)), R(t,g(s,t))}$$

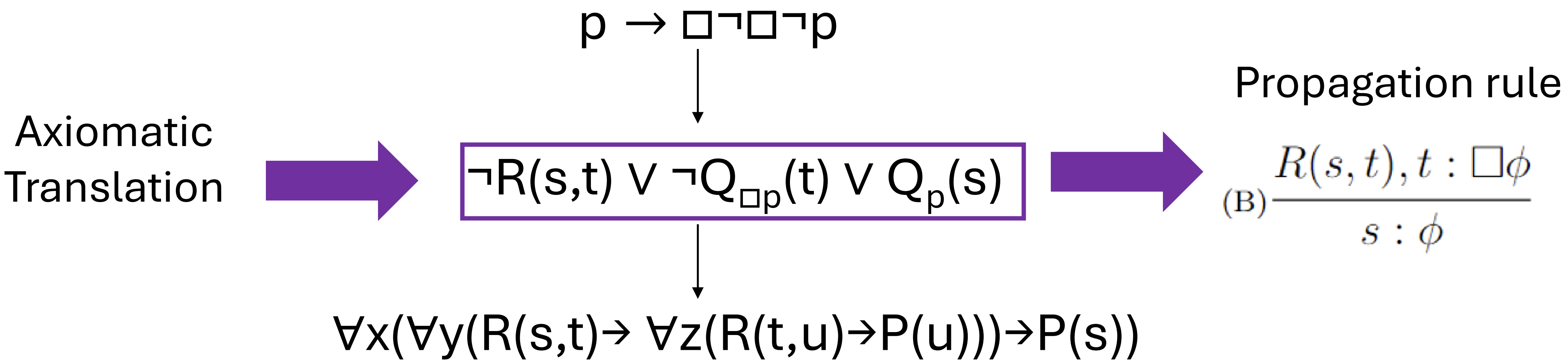


Leads to an infinite increase in labels



Idea of the method of the axiomatic translation principle and developing systems based on propagation rules

- Partially translate modal axiom B into a first-order formula and then a rule:



$$p \rightarrow \Box\neg\Box\neg p$$

Axiomatic Translation

$$\boxed{\neg R(s,t) \vee \neg Q_{\Box p}(t) \vee Q_p(s)}$$

Propagation rule

$$(\mathrm{B})\,\frac{R(s,t), t : \Box\phi}{s : \phi}$$

$$\forall x(\forall y(R(s,t) \rightarrow \forall z(R(t,u) \rightarrow P(u))) \rightarrow P(s))$$

Challenges with developing propagation rules

$$\text{(G0111.1)}\ \frac{R(s,t),\ t : \Box\phi}{s : \Diamond\phi} \qquad \text{(G0111.3)}\ \frac{R(s,t),\ t : \Box\psi}{t : \Diamond\psi}$$

$$\text{(G0111.2)}\ \frac{R(s,t),\ s : \Box\psi}{t : \Diamond\psi}$$

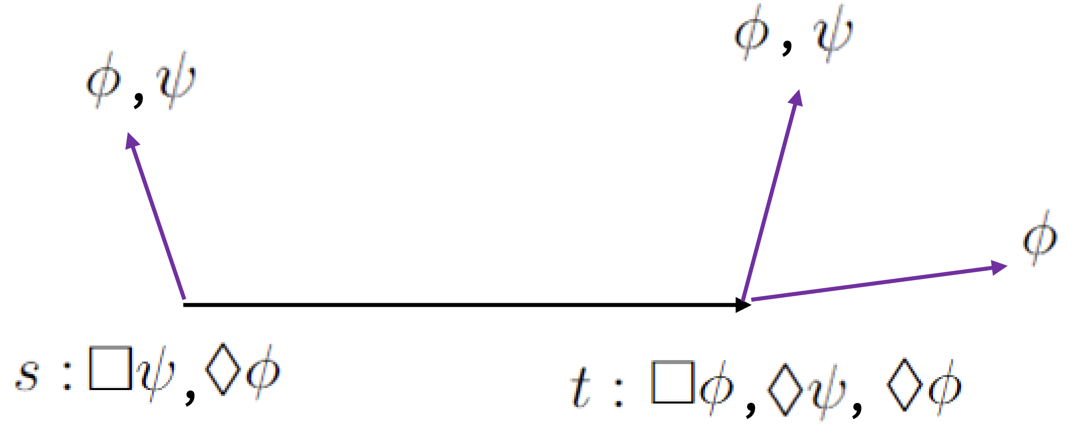
G0111

$$\Box\phi \rightarrow \Box\Diamond\phi$$

$$\Diamond\Box\phi \rightarrow \Diamond\phi$$

0,1,1,1-Convergent

$$\forall x, y(R(x,y) \rightarrow \exists z(R(x,z) \wedge R(y,z)))$$



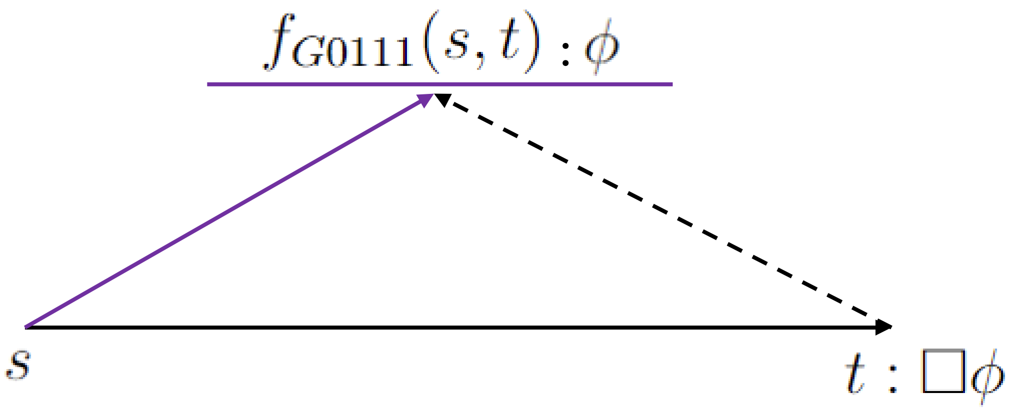
- This system is proved sound and complete with maxiscoping
- **Problems**: creates duplicate worlds which increases number of inferences, not generalisable (no systems for KG and KDe using these ideas)

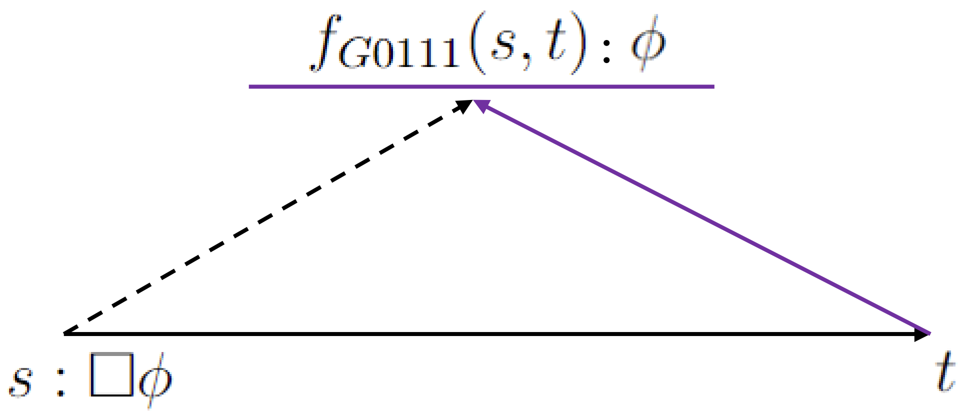
New method for developing refined rules

- **Idea**: instead refine the classical systems for the logics KG0111
- New refined system for KG0111

$$\underline{\text{(G0111.1)}}\ \frac{R(s,t),t:\Box\phi}{R(s,f_{G0111}(s,t)),f_{G0111}(s,t):\phi} \qquad \text{(G0111.2)}\ \frac{R(s,f_{G0111}(s,t)),f_{G0111}(s,t):\Box\phi}{R(t,f_{G0111}(s,t))}$$

$$\underline{\text{(G0111.3)}}\ \frac{R(s,t),s:\Box\phi}{R(t,f_{G0111}(s,t)),f_{G0111}(s,t):\phi} \qquad \text{(G0111.4)}\ \frac{R(t,f_{G0111}(s,t)),f_{G0111}(s,t):\Box\phi}{R(s,f_{G0111}(s,t))}$$

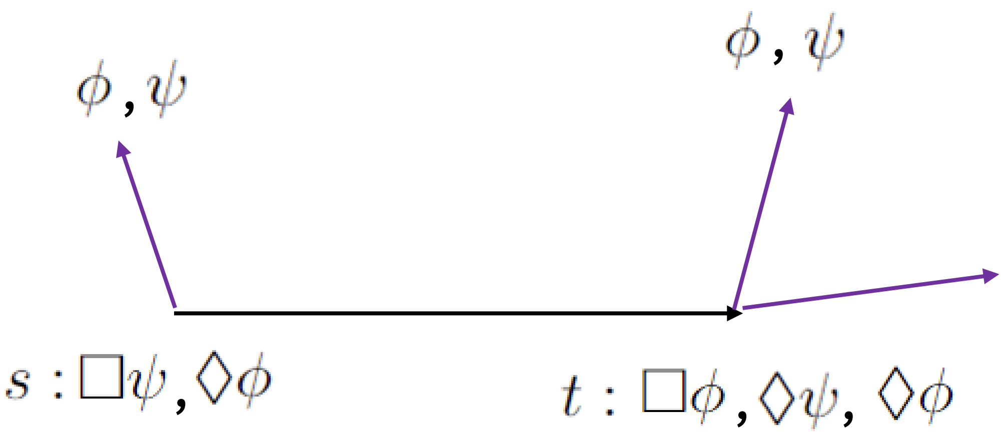


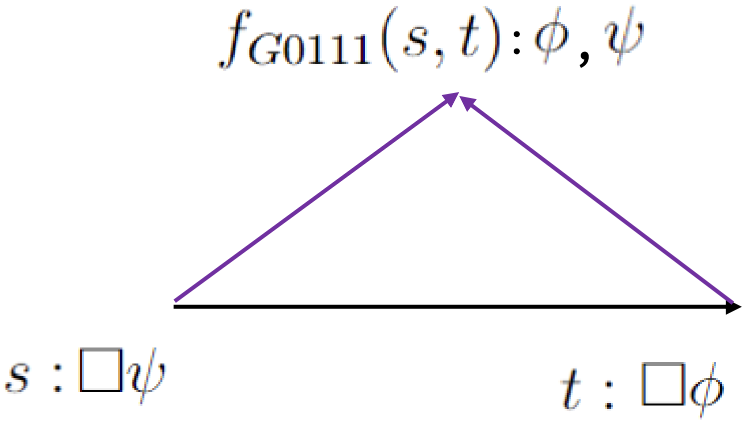


KG0111 example: revisited

$$(\mathrm{G0111.1})\ \frac{R(s,t),t:\Box\phi}{R(s,f_{G0111}(s,t)),f_{G0111}(s,t):\phi} \qquad (\mathrm{G0111.2})\ \frac{R(s,f_{G0111}(s,t)),f_{G0111}(s,t):\Box\phi}{R(t,f_{G0111}(s,t))}$$

$$(\mathrm{G0111.3})\ \frac{R(s,t),s:\Box\phi}{R(t,f_{G0111}(s,t)),f_{G0111}(s,t):\phi} \qquad (\mathrm{G0111.4})\ \frac{R(t,f_{G0111}(s,t)),f_{G0111}(s,t):\Box\phi}{R(s,f_{G0111}(s,t))}$$



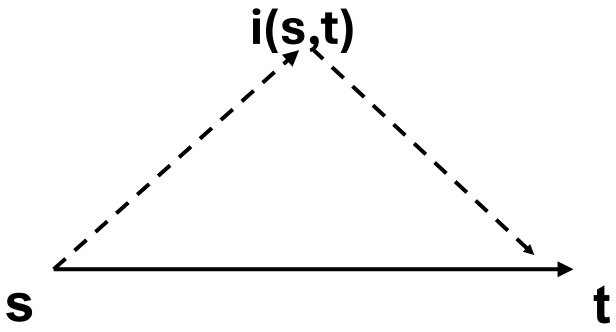


Refined tableau system for KDe

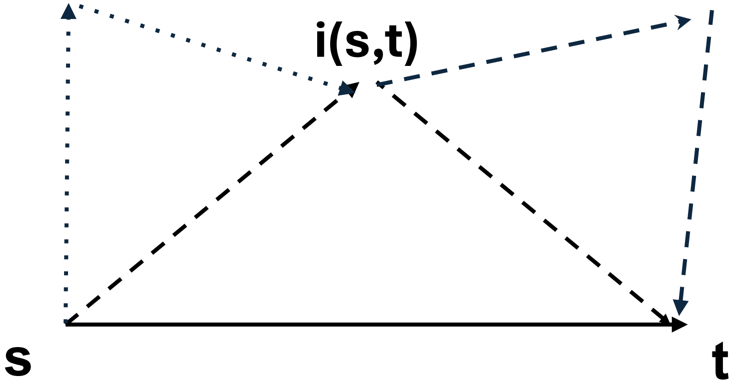
Density

$$\forall x, y(R(x,y) \to \exists z(R(x,z) \wedge R(z,y)))$$

$$\text{(dens)}\ \frac{R(s,t)}{R(s,i(s,t)), R(i(s,t),t)}$$



- Similar to KG0111, leads to infinite number of labels



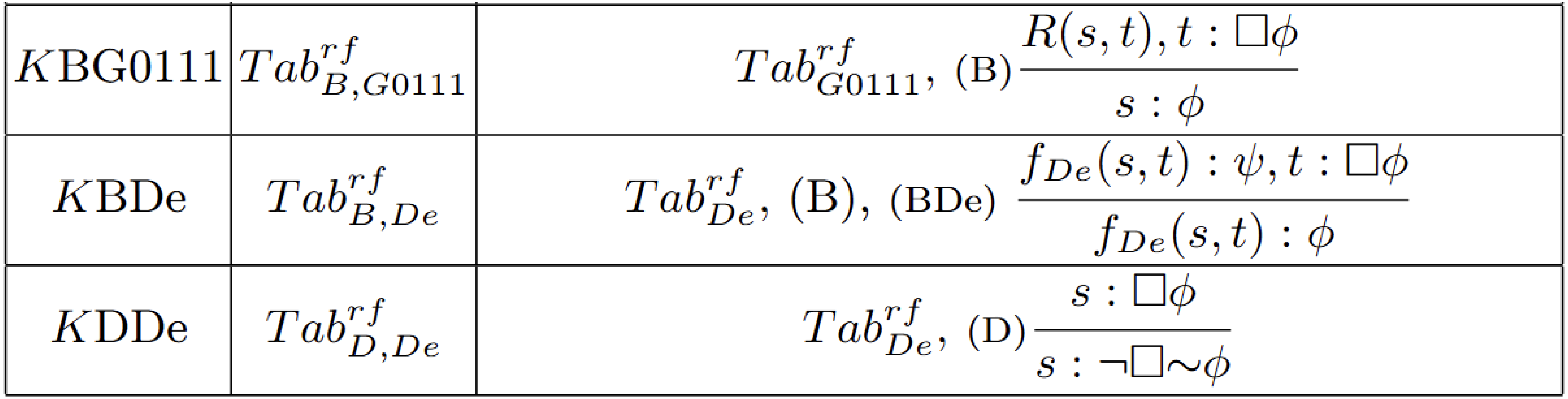
- **Our solution:** Refined system making use of unique label and labelled formulae for KDe tableau derivations

$$\text{(De.1)}\ \frac{R(s,t), s : \Box\phi, t : \Box\psi}{f_{De}(s,t) : \phi}$$

$$\text{(De.2)}\ \frac{f_{De}(s,t) : \Box\phi}{f_{De}(s,t) : \phi, t : \phi}$$

Results

- Refined tableau systems for the logics KG0111, KG, KDe
- Refined tableau systems for combinations:



KBG0111	$Tab^{rf}_{B,G0111}$	Tab^{rf}_{G0111}, (B) $\dfrac{R(s,t), t : \Box\phi}{s : \phi}$
KBDe	$Tab^{rf}_{B,De}$	Tab^{rf}_{De}, (B), (BDe) $\dfrac{f_{De}(s,t) : \psi, t : \Box\phi}{f_{De}(s,t) : \phi}$
KDDe	$Tab^{rf}_{D,De}$	Tab^{rf}_{De}, (D) $\dfrac{s : \Box\phi}{s : \neg\Box{\sim}\phi}$

Results (continued)

- Soundness and completeness results
- The systems and proofs are modular
- Challenge: show pre-models can be extended to concrete models
- Although models obtained are not concrete models, they can be extended to concrete models

Future Work

- Development of refined tableau systems for other logics in the large class of modal logics of confluence
- Termination results
- Implementation

Thank you

Questions?