The Modal Cube Revisited: Semantics without worlds

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Modal Logic

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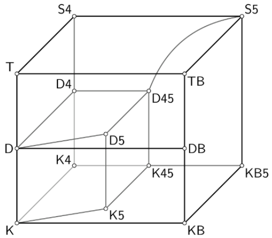
D $\Box A \to \Diamond A$

T $\Box A \to A$

B $A \to \Box\Diamond A$

4 $\Box A \to \Box\Box A$

5 $\Diamond A \to \Box\Diamond A$



Kripke Semantics

- The usual relational semantics of "possible worlds"

$$\mathcal{M}, w \Vdash \Box A \qquad \text{iff} \qquad \text{for all } wRv \text{ implies } \mathcal{M}, v \Vdash A;$$

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What about the "standard" truth table semantics?

Bad news

- Gödel: intuitionistic logic admits no finite-valued truth-functional semantics
- Since IPL can be faithfully embedded in S4, then S4 itself is not finite-valued.
- Dugundji: The above (negative) result holds for the whole modal cube.

We are then in a sort of dead end...

Kearns Semantics [Kea81]

- No need of possible worlds to give meaning to modalities.
- Non-deterministic matrix (nmatrix) generalize truth tables [AL05].
- 4 truth values for a complete system for KT, S4 and S5.

A	$\Box^{\mathsf{KT4}} A$	$\Diamond^{\mathsf{KT4}} A$
F	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
f	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{T}, \mathbf{t}\}$
t	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{T}, \mathbf{t}\}$
T	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$

A simple example in S4

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T	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$

$\alpha \to \beta$	**F**	**f**	**t**	**T**
F	**T**	**T**	**T**	**T**
f	**t**	**T**, **t**	**T**, **t**	**T**
t	**f**	**f**	**T**, **t**	**T**
T	**F**	**f**	**t**	**T**

- $p \to p$

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F	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
f	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{T}, \mathbf{t}\}$
t	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{T}, \mathbf{t}\}$
T	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$

$\alpha \rightarrow \beta$	**F**	**f**	**t**	**T**
F	**T**	**T**	**T**	**T**
f	**t**	$\mathbf{T}, \mathbf{t}$	$\mathbf{T}, \mathbf{t}$	**T**
t	**f**	**f**	$\mathbf{T}, \mathbf{t}$	**T**
T	**F**	**f**	**t**	**T**

- $p \rightarrow p$
- $\Diamond(p \rightarrow p)$

A simple example in S4

A	$\Box^{\mathrm{KT4}}A$	$\Diamond^{\mathrm{KT4}}A$
F	{**F**}	{**F**}
f	{**F**, **f**}	{**T**, **t**}
t	{**F**, **f**}	{**T**, **t**}
T	{**T**}	{**T**}

$\alpha \to \beta$	**F**	**f**	**t**	**T**
F	**T**	**T**	**T**	**T**
f	**t**	**T**, **t**	**T**, **t**	**T**
t	**f**	**f**	**T**, **t**	**T**
T	**F**	**f**	**t**	**T**

- $p \to p$
- $\Diamond(p \to p)$
- $\Box(p \to p)$ (bad surprise...)

Kearns Semantics [Kea81]

Soundness fails!

p	$p \to p$	$\Box(p \to p)$
F	**T**	**T**
f	**T**	**T**
f	**t**	**F, f**
t	**T**	**T**
t	**t**	**F, f**
T	**T**	**T**

- Kearns' solution: level valuations, that remove "undesirable" valuations.

Kearns Semantics [Kea81]

	p	$p \to p$	$\Box(p \to p)$	$\cdots$
	F	**T**	**T**	$\cdots$
	f	**T**	**T**	$\cdots$
$\times$	**f**	**t**	**F**, **f**	$\cdots$
	t	**T**	**T**	$\cdots$
$\times$	**t**	**t**	**F**, **f**	$\cdots$
	T	**T**	**T**	$\cdots$

- Since $p \to p$ is a tautology, a good level valuation must assign a designed value to $\Box\alpha$.
- This enforces the necessitation rule.

Kearns Semantics [Kea81]

	p	$p \to p$	$\Box(p \to p)$	$\cdots$
	F	**T**	**T**	$\cdots$
	f	**T**	**T**	$\cdots$
$\times$	**f**	**t**	**F, f**	$\cdots$
	t	**T**	**T**	$\cdots$
$\times$	**t**	**t**	**F, f**	$\cdots$
	T	**T**	**T**	$\cdots$

- Since $p \to p$ is a tautology, a good level valuation must assign a designed value to $\Box\alpha$.
- This enforces the necessitation rule.

However, this is NOT a decision procedure:

1. It requires to check all the tautologies.
2. Some rows will be removed "later" (when do we stop?)

Grätz Procedure [Grä22]: Partial Valuations

- Sound and complete decision procedure for KT and KT4.
- Only subformulas of the formula are evaluated.
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- Only subformulas of the formula are evaluated.
- Certain values creates dependencies that must be satisfied.
- E.g., **t** below is not properly supported:

	p	$p \to p$	$\Box(p \to p)$
	F	**T**	**T**
	f	**T**	**T**
×	**f**	**t**	**F**, **f**
	t	**T**	**T**
×	**t**	**t**	**F**, **f**
	T	**T**	**T**

Our Contribution

State of the art

- Kearns' level valuations for $9/15$ modal logics.
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- Kearns' level valuations and decision procedures for all the 15 logics.
- All such procedures are systematically constructed (and previous ones are obtained as instances).
- The key point: meaning and classification of truth values.

Ecumenism

The present [Kearnsean] semantic account is simpler than the standard [Kripkean] account [...] For I do not think there are such things as possible worlds, or even that they constitute a useful fiction.” (Kearns)

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Both semantics are indeed very well related!

Our relational model, on partial valuations, preserves the usual frame conditions in modal logics!

Outline

The Meaning of Truth Values

Level Valuations

Partial Valuations and Relational Model

Concluding Remarks

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Concluding Remarks

How Many Truth Values?

The 8 values introduced in [OS16] for K.

Truth-value	Intuitive meaning
$v(\alpha) = \mathbf{F}$	$\Diamond\neg\alpha \wedge \neg\alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathbf{f}$	$\Diamond\neg\alpha \wedge \neg\alpha \wedge \Diamond\alpha$
$v(\alpha) = \mathbf{f_1}$	$\Box\alpha \wedge \neg\alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathbf{f_2}$	$\Box\alpha \wedge \neg\alpha \wedge \Diamond\alpha$
$v(\alpha) = \mathbf{t_2}$	$\Diamond\neg\alpha \wedge \alpha \wedge \Box\neg\alpha$
$v(\alpha) = \mathbf{t_1}$	$\Box\alpha \wedge \alpha \wedge \Box\neg\alpha$
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Value function: $\mathbf{t}(\beta) = \Diamond\neg\beta \wedge \beta \wedge \Diamond\beta$

Distinguished Sets

Classification of the 8 values:

1. $\mathcal{D} = \{\mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2}\}$ (α is true)
2. $\mathcal{N} = \{\mathbf{T}, \mathbf{t_1}, \mathbf{f_2}, \mathbf{f_1}\}$ (α is necessary)
3. $\mathcal{I} = \{\mathbf{F}, \mathbf{f_1}, \mathbf{t_2}, \mathbf{t_1}\}$ ($\neg\alpha$ is necessary)
4. $\mathcal{P} = \{\mathbf{T}, \mathbf{t}, \mathbf{f_2}, \mathbf{f}\}$ (α is possible)
5. $\mathcal{PN} = \{\mathbf{F}, \mathbf{f}, \mathbf{t}, \mathbf{t_2}\}$ ($\neg\alpha$ is possible)

For example, $\mathbf{t_2}(\alpha) = \Diamond\neg\alpha \wedge \alpha \wedge \Box\neg\alpha$, hence:

- $\mathbf{t_2} \in \mathcal{PN}$ ($\neg\alpha$ is possible)
- $\mathbf{t_2} \in \mathcal{D}$ (designated value, α is true "now")
- $\mathbf{t_2} \in \mathcal{I}$ (α is impossible)

Do we need all the 8 values?

$\mathbf{t_1}$ and $\mathbf{f_1}$ denote "states" without successors:

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$$\mathbf{f_1}(\alpha) = \Box\neg\alpha \wedge \neg\alpha \wedge \Box\alpha$$

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- Not needed in logics characterizing serial frames.
- Our approach: The “modal characterization” of truth values yields conditions on those values. These conditions are systemically obtained for all the 15 logics.

Values per Family of Logics

Truth-value	Meaning
$v(\alpha) = \mathbf{F}$	$\Diamond\neg\alpha, \neg\alpha, \Box\neg\alpha$
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- $\mathcal{I} = \{\mathbf{F}, \mathbf{f_1}, \mathbf{t_2}, \mathbf{t_1}\}$
- $\mathcal{P} = \{\mathbf{T}, \mathbf{t}, \mathbf{f_2}, \mathbf{f}\}$
- $\mathcal{PN} = \{\mathbf{F}, \mathbf{f}, \mathbf{t}, \mathbf{t_2}\}$

Axiom		Condition	Rule
D	$\Box\alpha \to \Diamond\alpha$	$v(\alpha) \in \mathcal{N}$	$v(\alpha) \in \mathcal{P}$
		$v(\alpha) \in \mathcal{I}$	$v(\alpha) \in \mathcal{PN}$

Values allowed

- $\mathcal{V}(\mathrm{K}) = \{\mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2}, \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2}\}$
- $\mathcal{V}(\mathrm{KD}) = \{\mathbf{T}, \mathbf{t}, \cancel{\mathbf{t_1}}, \mathbf{t_2}, \mathbf{F}, \mathbf{f}, \cancel{\mathbf{f_1}}, \mathbf{f_2}\}$

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- $\mathcal{I} = \{\mathbf{F}, \mathbf{f_1}, \mathbf{t_2}, \mathbf{t_1}\}$
- $\mathcal{P} = \{\mathbf{T}, \mathbf{t}, \mathbf{f_2}, \mathbf{f}\}$
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Axiom		Condition	Rule
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		$v(\alpha) \in \mathcal{I}$	$v(\alpha) \in \mathcal{PN}$
T	$\Box\alpha \to \alpha$	$v(\alpha) \in \mathcal{N}$	$v(\alpha) \in \mathcal{D}$
		$v(\alpha) \in \mathcal{I}$	$v(\alpha) \notin \mathcal{D}$

Values allowed

- $\mathcal{V}(\mathrm{K}) = \{\mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2}, \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2}\}$
- $\mathcal{V}(\mathrm{KD}) = \{\mathbf{T}, \mathbf{t}, \cancel{\mathbf{t_1}}, \mathbf{t_2}, \mathbf{F}, \mathbf{f}, \cancel{\mathbf{f_1}}, \mathbf{f_2}\}$
- $\mathcal{V}(\mathrm{KT}) = \{\mathbf{T}, \mathbf{t}, \cancel{\mathbf{t_1}}, \cancel{\mathbf{t_2}}, \mathbf{F}, \mathbf{f}, \cancel{\mathbf{f_1}}, \cancel{\mathbf{f_2}}\}$

The Matrices

$\alpha \to \beta$	F	f	f_1	f_2	t_2	t_1	t	T
F	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$
f	$\{t\}$	$\{T, t\}$	$\{t_1\}$	$\{T\}$	$\{t\}$	$\{T\}$	$\{T, t\}$	$\{T\}$
f_1	$\{t_2\}$	$\{t\}$	$\{t_1\}$	$\{T\}$	$\{t_2\}$	$\{t_1\}$	$\{t\}$	$\{T\}$
f_2	$\{t_2\}$	$\{t\}$	$\{t_1\}$	$\{T\}$	$\{t_2\}$	$\{t_1\}$	$\{t\}$	$\{T\}$
t_2	$\{f_2\}$	$\{f_2\}$	$\{f_2\}$	$\{f_2\}$	$\{T\}$	$\{T\}$	$\{T\}$	$\{T\}$
t_1	$\{F\}$	$\{f\}$	$\{f_1\}$	$\{f_2\}$	$\{t_2\}$	$\{t_1\}$	$\{t_2\}$	$\{T\}$
t	$\{f\}$	$\{f, f_2\}$	$\{f_2\}$	$\{f_2\}$	$\{t\}$	$\{T\}$	$\{T, t\}$	$\{T\}$
T	$\{F\}$	$\{f\}$	$\{f_1\}$	$\{f_2\}$	$\{t_2\}$	$\{t_1\}$	$\{t\}$	$\{T\}$

The Matrices

α	$\Box^{K}\alpha$	$\Box^{KB}\alpha$	$\Box^{K4}\alpha$	$\Box^{K5}\alpha$	$\Box^{K45}\alpha$
$\mathbf{F}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\mathbf{f}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\mathbf{f_1}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$
$\mathbf{f_2}$	$\{\mathbf{T}, \mathbf{t}, \mathbf{t_2}\}$	$\{\mathbf{t_2}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T}, \mathbf{t_2}\}$	$\{\mathbf{T}\}$
$\mathbf{t_2}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\mathbf{t_1}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$	$\{\mathbf{t_1}\}$
$\mathbf{t}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}, \mathbf{f}, \mathbf{f_2}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
$\mathbf{T}$	$\{\mathbf{T}, \mathbf{t}, \mathbf{t_2}\}$	$\{\mathbf{T}, \mathbf{t}, \mathbf{t_2}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T}, \mathbf{t_2}\}$	$\{\mathbf{T}\}$

The Matrices

α	$\Box^{\mathsf{KT}}\alpha$	$\Box^{\mathsf{KTB}}\alpha$	$\Box^{\mathsf{KT4}}\alpha$	$\Box^{\mathsf{KTB45}}\alpha$
F	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}\}$
f	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{F}\}$	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{F}\}$
t	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{F}, \mathbf{f}\}$	$\{\mathbf{F}\}$
T	$\{\mathbf{T}, \mathbf{t}\}$	$\{\mathbf{T}, \mathbf{t}\}$	$\{\mathbf{T}\}$	$\{\mathbf{T}\}$

The Matrices

α	$\square^{KD}\alpha$	$\square^{KDB}\alpha$	$\square^{KD4}\alpha$	$\square^{KD5}\alpha$	$\square^{KD45}\alpha$
F	{**F**, **f**, **f**$_2$}	{**F**}	{**F**}	{**F**}	{**F**}
f	{**F**, **f**, **f**$_2$}	{**F**}	{**F**, **f**, **f**$_2$}	{**F**}	{**F**}
f$_2$	{**T**, **t**, **t**$_2$}	{**t**$_2$}	{**T**}	{**T**, **t**$_2$}	{**T**}
t$_2$	{**F**, **f**, **f**$_2$}	{**F**, **f**, **f**$_2$}	{**F**}	{**F**}	{**F**}
t	{**F**, **f**, **f**$_2$}	{**F**, **f**, **f**$_2$}	{**F**, **f**, **f**$_2$}	{**F**}	{**F**}
T	{**T**, **t**, **t**$_2$}	{**T**, **t**, **t**$_2$}	{**T**}	{**T**, **t**$_2$}	{**T**}

Outline

The Meaning of Truth Values

Level Valuations

Partial Valuations and Relational Model

Concluding Remarks

Level Semantics Revisited

Definition (Level valuation in $\mathcal{M}_\star$)

Let $Val(\mathcal{M}_\star)$ be the set of valuation functions in $\mathcal{M}_\star$.

$\mathcal{L}_0(\mathcal{M}_\star)$ Every $v \in Val(\mathcal{M}_\star)$ where, if $\exists \alpha. v(\alpha) \in \{\mathbf{t_1}, \mathbf{f_1}\}$, then $\forall \beta, v(\beta) \in \{\mathbf{t_1}, \mathbf{f_1}\}$.

$\mathcal{L}_{k+1}(\mathcal{M}_\star)$ Every $v \in \mathcal{L}_k$ such that, for every formula α, if $\models^{\mathcal{L}_k} \alpha$, then $v(\alpha) \in \{\mathbf{T}, \mathbf{t_1}\}$

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$\mathcal{L}_{k+1}(\mathcal{M}_\star)$ Every $v \in \mathcal{L}_k$ such that, for every formula α, if $\models^{\mathcal{L}_k} \alpha$, then $v(\alpha) \in \{\mathbf{T}, \mathbf{t_1}\}$

The set of level valuations in $\mathcal{M}_\star$ is given by

$$\mathcal{L}(\mathcal{M}_\star) = \bigcap_{n=0}^{\infty} \mathcal{L}_n$$

Soundness ($\Gamma \vdash^\star \alpha \Rightarrow \Gamma \models^{\mathcal{L}(\mathcal{M}_\star)} \alpha$) is easy.

Completeness of Level Valuations

- Henkin construction where characteristic functions are obtained directly from the meaning of the truth values.

$$v^{\mathcal{L}}_{\Delta}(\alpha) = \iota \text{ iff } \Delta \vdash^{\mathcal{L}} \iota(\alpha)$$

Completeness of Level Valuations

- Henkin construction where characteristic functions are obtained directly from the meaning of the truth values.

$$v^{\mathcal{L}}_{\Delta}(\alpha) = \iota \text{ iff } \Delta \vdash^{\mathcal{L}} \iota(\alpha)$$

For instance, for the family KT$\star$:

$$v^{\mathcal{L}}_{\Delta}(\alpha) = \begin{cases} \mathbf{F} & \text{iff } \Delta \vdash^{\mathcal{L}} \Box\neg\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \neg\alpha \wedge \Diamond\neg\alpha) \\ \mathbf{f} & \text{iff } \Delta \vdash^{\mathcal{L}} \neg\alpha \wedge \Diamond\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \Diamond\neg\alpha) \\ \mathbf{t} & \text{iff } \Delta \vdash^{\mathcal{L}} \alpha \wedge \Diamond\neg\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \Diamond\alpha) \\ \mathbf{T} & \text{iff } \Delta \vdash^{\mathcal{L}} \Box\alpha \text{ (and } \Delta \vdash^{\mathcal{L}} \alpha \wedge \Diamond\alpha) \end{cases}.$$

Lemma (Adequacy)

For every logic $\mathcal{L}$ and maximally consistent set Δ, $v^{\mathcal{L}}_{\Delta}$ is a level valuation.

Theorem (Completeness)

For every modal logic $\mathcal{L}$ and associated Nmatrix $\mathcal{M}$,
$\Gamma \vDash^{\mathcal{L}(\mathcal{M}_\star)} \alpha \Rightarrow \Gamma \vdash^{\star} \alpha$.

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Back to the Meaning of Values

Consider a valuation v s.t $v(\alpha) = \mathbf{f_2}$:

- Recall: $\mathbf{f_2}(\alpha) = \Box\alpha \wedge \neg\alpha \wedge \Diamond\alpha$.
- Hence, $\Diamond\alpha$ needs to be true.
- This requires the existence of a valuation v' s.t. $v'(\alpha) \in \mathcal{D}$, thus fulfilling the requirement $\Diamond\alpha$.

Back to the Meaning of Values

Consider a valuation v s.t $v(\alpha) = \mathbf{f_2}$:

- Recall: $\mathbf{f_2}(\alpha) = \Box\alpha \wedge \neg\alpha \wedge \Diamond\alpha$.
- Hence, $\Diamond\alpha$ needs to be true.
- This requires the existence of a valuation v' s.t. $v'(\alpha) \in \mathcal{D}$, thus fulfilling the requirement $\Diamond\alpha$.
- Such v' must satisfy some extra requirements (due to $\Box$):
  - By NEC: if $v(\beta) \in \mathcal{N}$ and vRv' then $v'(\beta) \in \mathcal{D}$
  - In, e.g., K4: if $v(\beta) \in \mathcal{N}$ and vRv' then $v'(\beta) \in \mathcal{N}$.

We will systematically build a relational model for partial valuations.

Relational Model

Definition (Pre-model $\langle \Pi, R \rangle$)

Where $\Pi \subseteq [\Lambda \to \mathcal{V}]_{\mathcal{M}}$ is a set of partial valuations, and $R \subseteq \Pi \times \Pi$ relates valuations:

1. If $v(\alpha) \in \mathcal{P}$, then $\exists v' \in \Pi$ such that vRv' and $v'(\alpha) \in \mathcal{D}$;
2. If $v(\alpha) \in \mathcal{PN}$, then $\exists v' \in \Pi$ such that vRv' and $v'(\alpha) \notin \mathcal{D}$.

$$
\begin{array}{rl}
 & \alpha \ldots \beta \ldots \\
v : & \ldots \mathcal{P} \ldots \mathcal{PN} \ldots \\
 & \downarrow\exists \quad \downarrow\exists \\
v' : & \ldots \mathcal{D} \ldots \mathcal{D}^{\complement} \ldots
\end{array}
$$

From Pre-models to K-Models

Property		Condition	Implies
nec		$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{D}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \notin \mathcal{D}$

- Notation $\iota \Rightarrow^R V$: if $v(\alpha) = \iota$ and vRv', then $v'(\alpha) \in V$.

$\mathbf{T} \Rightarrow^R \mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2}$

$$
\begin{array}{rl}
 & \alpha \ldots \beta \ldots \\
v: & \ldots \mathbf{T} \ldots \mathcal{P}, \mathcal{PN} \ldots \\
 & \Downarrow_R \quad \downarrow_\exists \\
v': & \ldots \mathcal{D} \ldots \mathcal{D}, \mathcal{D}^{\complement} \ldots
\end{array}
$$

$\mathbf{F} \Rightarrow^R \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2}$

$$
\begin{array}{rl}
 & \alpha \ldots \quad \beta \ldots \\
v: & \ldots \mathbf{F} \ldots \quad \mathcal{P}, \mathcal{PN} \ldots \\
 & \Downarrow_R \quad \downarrow_\exists \\
v': & \ldots \mathcal{D}^{\complement} \ldots \mathcal{D}, \mathcal{D}^{\complement} \ldots
\end{array}
$$

From K-Models to K4$\star$ Models

Property		Condition	Implies
4	$\Box\alpha \to \Box\Box\alpha$	$v(\alpha) \in \mathcal{N}$ and vRv'	$v'(\alpha) \in \mathcal{N}$
		$v(\alpha) \in \mathcal{I}$ and vRv'	$v'(\alpha) \in \mathcal{I}$

K	K4
$\mathbf{T} \Rightarrow^R \mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2}$	$\mathbf{T} \Rightarrow^R \mathbf{T}, \mathbf{t_1}$
$\mathbf{t_2} \Rightarrow^R \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2}$	$\mathbf{t_2} \Rightarrow^R \mathbf{F}, \mathbf{f_1}$

Distinguished sets

- $\mathcal{N} = \{\mathbf{T}, \mathbf{t_1}, \mathbf{f_2}, \mathbf{f_1}\}$
- $\mathcal{I} = \{\mathbf{F}, \mathbf{f_1}, \mathbf{t_2}, \mathbf{t_1}\}$

The Whole Picture (The Recipe)

Property		Condition	Implies
nec		$w(\alpha) \in \mathcal{N}$ and wRw'	$w'(\alpha) \in \mathcal{D}$
		$w(\alpha) \in \mathcal{I}$ and wRw'	$w'(\alpha) \notin \mathcal{D}$
t	$\Box\alpha \to \alpha$	$w(\alpha) \in \mathcal{N}$	$w(\alpha) \in \mathcal{D}$
		$w(\alpha) \in \mathcal{I}$	$w(\alpha) \notin \mathcal{D}$
d	$\Box\alpha \to \Diamond\alpha$	$w(\alpha) \in \mathcal{N}$	$w(\alpha) \in \mathcal{P}$
		$w(\alpha) \in \mathcal{I}$	$w(\alpha) \in \mathcal{PN}$
b	$\alpha \to \Box\Diamond\alpha$	$w(\alpha) \in \mathcal{D}$ and wRw'	$w'(\alpha) \in \mathcal{P}$
		$w(\alpha) \notin \mathcal{D}$ and wRw'	$w'(\alpha) \in \mathcal{PN}$
4	$\Box\alpha \to \Box\Box\alpha$	$w(\alpha) \in \mathcal{N}$ and wRw'	$w'(\alpha) \in \mathcal{N}$
		$w(\alpha) \in \mathcal{I}$ and wRw'	$w'(\alpha) \in \mathcal{I}$
5	$\Diamond\alpha \to \Box\Diamond\alpha$	$w(\alpha) \in \mathcal{P}$ and wRw'	$w'(\alpha) \in \mathcal{P}$
		$w(\alpha) \in \mathcal{PN}$ and wRw'	$w'(\alpha) \in \mathcal{PN}$
	$\Box\Box\alpha \to \Box\Box\Box\alpha$	$w(\alpha), w'(\alpha) \in \mathcal{N}$, wRw' and (wRw'' or $w'Rw''$)	$w''(\alpha) \in \mathcal{N}$
		$w(\alpha), w'(\alpha) \in \mathcal{I}$, wRw' and (wRw'' or $w'Rw''$)	$w''(\alpha) \in \mathcal{I}$

The Whole Picture (The Dishes)

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_1}, \mathbf{t_2} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_1}, \mathbf{f_2}
\end{array}$$

(a) K

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{f}, \mathbf{f_2} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{f}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{t}, \mathbf{t_2} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{t}, \mathbf{t_2} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(b) KB

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t_1} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f_1} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t_1} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f_1}
\end{array}$$

(c) K4

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(d) K5

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T} \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(e) K45

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_1} & \Rightarrow^R & \bullet \\
\mathbf{f_1} & \Rightarrow^R & \bullet \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(f) KB45

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_2} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{t_2} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}, \mathbf{f_2}
\end{array}$$

(g) KD

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{f}, \mathbf{f_2} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{f}, \mathbf{f_2} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{t}, \mathbf{t_2} \\
\mathbf{f} & \Rightarrow^R & \mathbf{F}, \mathbf{t}, \mathbf{f}, \mathbf{t_2} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(h) KDB

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(i) KD4

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F}, \mathbf{f} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(j) KD5

$$\begin{array}{lll}
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{t_2} & \Rightarrow^R & \mathbf{F} \\
\mathbf{f_2} & \Rightarrow^R & \mathbf{T} \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(k) KD45

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{F}, \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{T}, \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(l) KT

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T}, \mathbf{t} \\
\mathbf{t} & \Rightarrow^R & \mathbf{T}, \mathbf{t}, \mathbf{f} \\
\mathbf{f} & \Rightarrow^R & \mathbf{F}, \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}, \mathbf{f}
\end{array}$$

(m) KTB

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{F}, \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{T}, \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(n) KT4

$$\begin{array}{lll}
\mathbf{t} & \rightarrowtail & \mathbf{f} \\
\mathbf{f} & \rightarrowtail & \mathbf{t} \\
\\
\mathbf{T} & \Rightarrow^R & \mathbf{T} \\
\mathbf{t} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{f} & \Rightarrow^R & \mathbf{t}, \mathbf{f} \\
\mathbf{F} & \Rightarrow^R & \mathbf{F}
\end{array}$$

(o) KTB45

Frame Properties

According to the logic $\mathcal{L}$, the relation induced by $\Rightarrow^R$ is serial/reflexive/symmetric/transitive/Euclidian.

Building Tables

- Models can be extended with "new columns"
- This procedure is deterministic.

Theorem (Analyticity (Procedure))

Every partial level-valuation can be extended to a level-valuation.

Theorem (Soundness)

For every $\mathcal{L}$, $\Gamma \vdash^{\mathcal{L}} \alpha \Rightarrow \Gamma \vDash^{\mathcal{L}} \alpha$.

Completeness of Partial Valuations

- We show that level valuations restricted to a (closed) domain are good partial valuations.
- This is called co-analyticity.
- The proof is entirely guided by the modal characterization of truth values.

Completeness of Partial Valuations

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- This is called co-analyticity.
- The proof is entirely guided by the modal characterization of truth values.

Theorem (Co-analyticity)

For every level-valuation v and every set closed under subformulas Λ*,* $v \downarrow_\Lambda$ *is a partial level-valuation.*

Theorem (Completeness)

For every $\mathcal{L}$*,* $\Gamma \models^{\mathcal{L}} \alpha \Rightarrow \Gamma \vdash^{\mathcal{L}} \alpha$.

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Concluding Remarks

Our nmatrices were computed (and refined) with the aid of a Rocq procedure (impossible by hand!). E.g, axiom **K** in KTB45:

Info Command Prooftree Interrupt Restart Help

```
     = ([! "p0"; ! "p1"; ! "p0" ~> ! "p1"; [] ! "p0"; [] ! "p1"; []
         (! "p0" ~> ! "p1"); [] ! "p0" ~> [] ! "p1";
         [] (! "p0" ~> ! "p1") ~> ([] ! "p0" ~> [] ! "p1")];
       [(12; [0; 0; 7; 0; 0; 7; 7; 7]); (24; [0; 1; 7; 0; 0; 7; 7; 7]);
        (36; [0; 6; 7; 0; 0; 7; 7; 7]); (47; [0; 7; 7; 0; 7; 7; 7; 7]);
        (59; [1; 0; 6; 0; 0; 0; 7; 7]); (71; [1; 1; 7; 0; 0; 7; 7; 7]);
        (83; [1; 1; 6; 0; 0; 0; 7; 7]); (95; [1; 6; 7; 0; 0; 7; 7; 7]);
        (107; [1; 6; 6; 0; 0; 0; 7; 7]); (118; [1; 7; 7; 0; 7; 7; 7; 7]);
        (130; [6; 0; 1; 0; 0; 0; 7; 7]); (142; [6; 1; 1; 0; 0; 0; 7; 7]);
        (154; [6; 6; 7; 0; 0; 7; 7; 7]); (166; [6; 6; 6; 0; 0; 0; 7; 7]);
        (177; [6; 7; 7; 0; 7; 7; 7; 7]); (188; [7; 0; 0; 7; 0; 0; 0; 7]);
        (199; [7; 1; 1; 7; 0; 0; 0; 7]); (210; [7; 6; 6; 7; 0; 0; 0; 7]);
        (222; [7; 7; 7; 7; 7; 7; 7; 7])])
     : pair (list LF) (list (pair nat (list nat)))
```

Future work

Full mechanization of our proofs. Partial results for S5.

Concluding Remarks

Showing analyticity correct for KD requires checking more than 15M cases!

Maude to the rescue:

```
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t]: (2 <- 1),
[A : f2, C : f2, NEW : t2]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : T]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : t]: (2 <- 1)))
16 : ok((
[A : t, C : f, NEW : f]: 1,
[A : t2, C : f2, NEW : f2]: (2 <- 1),
[A : f2, C : t2, NEW : t2]: (2 <- 1),
[A : f2, C : f2, NEW : t2]: (2 <- 1)))
```

Future Work

- Intuitionistic modal cube: combining the nmatrices for LJ in [LCL24] with those proposed here.
- Non-normal modalities? How many values are needed?
- Counter-models (some partial results with our Rocq tool)
- Relating complexity results?
- NMatrices for Ecumenical systems (already in progress).

Thank you!

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