Non-Wellfounded Proof Theory for Interpretability Logic

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TABLEAUX 2025
Reykjavík, September 29

Interpretability logic IL

- **Interpretability logic**: a modal logic corresponding to the notion of relative interpretability between first-order theories.
- **Syntax of interpretability logic**: given by

$$\phi ::= p \mid \bot \mid \phi \to \phi \mid \phi \rhd \phi,$$

  where p ranges over a fixed set of propositional variables
- We treat other Boolean connectives and modal operators $\Box$ and $\Diamond$ as abbreviations. In particular, the usual modalities are defined as:

$$\Box\phi := \neg\phi \rhd \bot, \qquad \Diamond\phi := \neg(\phi \rhd \bot).$$

Hilbert axiomatization of IL

The Hilbert-style axiomatization of IL is given by the following axioms:

- tautologies of classical propositional logic,

(K) $\Box(\phi \to \psi) \to (\Box\phi \to \Box\psi)$,

(4) $\Box\phi \to \Box\Box\phi$,

(L) $\Box(\Box\phi \to \phi) \to \Box\phi$,

(J1) $\Box(\phi \to \psi) \to (\phi \rhd \psi)$,

(J2) $(\phi \rhd \chi) \wedge (\chi \rhd \psi) \to (\phi \rhd \psi)$,

(J3) $(\phi \rhd \psi) \wedge (\chi \rhd \psi) \to (\phi \vee \chi) \rhd \psi$,

(J4) $\phi \rhd \psi \to (\Diamond\phi \to \Diamond\psi)$,

(J5) $\Diamond\phi \rhd \phi$,

and inference rules modus ponens and necessitation:

$$\frac{\phi \to \psi \qquad \phi}{\psi}\,, \qquad \frac{\phi}{\Box\phi}\,.$$

Sasaki's work on IL and variations

- Cut-free systems for logics IL and IK4 (axioms of IL without the L axiom):

  Katsumi Sasaki, *A cut-free sequent system for the smallest interpretability logic*, Studia Logica, 70(3):353–372, 2002.

- Another cut-free system for the logic IK4:

  Katsumi Sasaki, *A sequent system for a sublogic of the smallest interpretability logic*, Journal of the Nanzan Academic Society, Mathematical Sciences and Information Engineering, 3:1–12, 2003.

- Cut-free system for logic IK4P and system for ILP (that is conjectured to be cut-free):

  Katsumi Sasaki, *A sequent system for the interpretability logic with the persistence axiom*, Journal of the Nanzan Academic Society, Mathematical Sciences and Information Engineering, 2:25–34, 2002.

Completeness via cut elimination for provability logics

$$
\begin{tikzcd}
\mathsf{IL} \arrow[r, shift left] & \mathcal{G}\mathsf{IL} + \mathrm{Cut} \arrow[l, shift left] \arrow[r, "\text{Löb}"] & \mathcal{G}^{\infty}\mathsf{IL} + \mathrm{Cut} \arrow[d, "\text{Cut elim}"] \\
 & \mathcal{G}\mathsf{IL} \arrow[u, "\text{Trivial}"] & \mathcal{G}^{\infty}\mathsf{IL} \arrow[l, "\text{Rec}"]
\end{tikzcd}
$$

Sequents

- A **sequent** is an expression of the form

$$\Gamma \Rightarrow \Delta$$

  where Γ and Δ are finite multisets of formulas
- We will use the following notation:

$$\Gamma \rhd \bot := \{\phi \rhd \bot \mid \phi \in \Gamma\}$$

$$\Phi_I := \{\phi_i \mid i \in I\}$$

- we write $[S_i]_{m \dots i \dots 0}$ for the finite sequence of sequents $(S_m, S_{m-1}, \dots, S_0)$

Wellfounded sequent calculus $\mathcal{G}$IL

We define the sequent calculus $\mathcal{G}$IL as the wellfounded calculus given by the following rules:

$$\dfrac{}{p, \Gamma \Rightarrow p, \Delta}\ \mathsf{ax} \qquad \dfrac{}{\bot, \Gamma \Rightarrow \Delta}\ \bot\mathsf{L}$$

$$\dfrac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \to \psi, \Gamma \Rightarrow \Delta}\ \to \mathsf{L} \qquad \dfrac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \to \psi}\ \to \mathsf{R}$$

$$\dfrac{[\psi_i, (\psi_i, \Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta}\ \rhd_{\mathsf{IL}}$$

The system $\mathcal{G}$IL $+\mathrm{Cut}$ also has the following Cut rule:

$$\dfrac{\Gamma \Rightarrow \Delta, \chi \qquad \chi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\ \mathrm{Cut}$$

The rule $\rhd_{\mathsf{IL}}$ - an example

Let $m = 2$. Then we have

$$\dfrac{S_2 \qquad S_1 \qquad S_0}{\phi_0 \rhd \psi_0, \phi_1 \rhd \psi_1, \Gamma \Rightarrow \psi_2 \rhd \phi, \Delta} \rhd_{\mathsf{IL}}$$

where

$$S_2 = \psi_2, (\psi_2, \phi_0, \phi_1, \phi) \rhd \bot \Rightarrow \phi_0, \phi_1, \phi$$
$$S_1 = \psi_1, (\psi_1, \phi_0, \phi) \rhd \bot \Rightarrow \phi_0, \phi$$
$$S_0 = \psi_0, (\psi_0, \phi) \rhd \bot \Rightarrow \phi$$

Proofs in $\mathcal{G}$IL

A wellfounded proof in $\mathcal{G}$IL $(+\mathrm{Cut})$ is a finite tree whose nodes are marked by sequents and that is constructed to the rules of $\mathcal{G}$IL $(+\mathrm{Cut})$.

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\phi, (\phi, \psi, \phi, \chi) \rhd \bot \Rightarrow \psi, \phi, \chi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\psi, (\psi, \phi, \chi) \rhd \bot \Rightarrow \phi, \chi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\chi, (\chi, \chi) \rhd \bot \Rightarrow \chi$}
\RightLabel{$\rhd_{\mathrm{IL}}$}
\TrinaryInfC{$\phi \rhd \psi, \psi \rhd \chi \Rightarrow \phi \rhd \chi$}
\end{prooftree}
$$

$$
\begin{prooftree}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\phi, (\neg\neg\phi, \phi, \bot) \rhd \bot \Rightarrow \bot, \phi$}
\RightLabel{$\neg$R}
\UnaryInfC{$(\neg\neg\phi, \phi, \bot) \rhd \bot \Rightarrow \bot, \phi, \neg\phi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\neg\phi, (\neg\neg\phi, \phi, \bot) \rhd \bot \Rightarrow \bot, \phi$}
\AxiomC{}
\RightLabel{$\bot$L}
\UnaryInfC{$\bot, (\bot, \bot) \rhd \bot \Rightarrow \bot$}
\RightLabel{$\rhd_{\mathrm{IL}}$}
\BinaryInfC{$(\Diamond\phi, \phi) \rhd \bot \Rightarrow \phi, \neg\neg\phi \rhd \bot$}
\RightLabel{$\neg$L}
\UnaryInfC{$\Diamond\phi, (\Diamond\phi, \phi) \rhd \bot \Rightarrow \phi$}
\RightLabel{$\rhd_{\mathrm{IL}}$}
\UnaryInfC{$\Rightarrow \Diamond\phi \rhd \phi$}
\end{prooftree}
$$

Hilbert style proofs in IL and sequent proofs in $\mathcal{G}$IL $+ \mathrm{Cut}$

- By induction on the length of the Hilbert proof of ϕ we can prove:

$$\text{if } IL \vdash \phi, \text{ then } \mathcal{G}\mathsf{IL} + \mathrm{Cut} \vdash \Rightarrow \phi.$$

- Let S be a sequent $\Gamma \Rightarrow \Delta$. We define the IL-formula $S^\sharp$ as the formula

$$\bigwedge \Gamma \to \bigvee \Delta.$$

Theorem

IL $\vdash S^\sharp$ *if and only if* $\mathcal{G}$IL $+ \mathrm{Cut} \vdash S$.

Non-wellfounded calculus

- Allows proofs with infinite branches.
- To guarantee that there is no vicious infinite reasoning, it is usual to add a constraint to the possible infinite paths in the proof, e.g. enforce that any infinite branch goes through the premise of a rule infinite often

A **local-progress sequent calculus** is a pair $\mathcal{G} = (\mathcal{R}, L)$ where:

(i) $\mathcal{R}$ is a set of rules, called the rules of $\mathcal{C}$,

(ii) L is a function that given a rule $R \in \mathcal{R}$ and an instance of the rule

$$\frac{S_0 \qquad \cdots \qquad S_{n-1}}{S}\, R$$

returns a subset of $\{0, \ldots, n-1\}$. This subset is the set of premises of the rule instance that make progress.

Sequent calculus $\mathcal{G}^\infty\mathsf{IL}$

We define the sequent calculus $\mathcal{G}^\infty\mathsf{IL}$ as the local-progress sequent calculus given by the following rules:

$$\frac{}{p, \Gamma \Rightarrow p, \Delta}\,\mathsf{ax} \qquad \frac{}{\bot, \Gamma \Rightarrow \Delta}\,\bot\mathsf{L}$$

$$\frac{\Gamma \Rightarrow \Delta, \phi \qquad \psi, \Gamma \Rightarrow \Delta}{\phi \to \psi, \Gamma \Rightarrow \Delta}\,\to\!\mathsf{L} \qquad \frac{\phi, \Gamma \Rightarrow \Delta, \psi}{\Gamma \Rightarrow \Delta, \phi \to \psi}\,\to\!\mathsf{R}$$

$$\frac{[\psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \ldots i \ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta}\,\rhd_{\mathsf{IK4}}$$

Progress only occurs at the premises of $\rhd_{\mathsf{IK4}}$. The system with the Cut rule:

$$\frac{\Gamma \Rightarrow \Delta, \chi \qquad \chi, \Gamma \Rightarrow \Delta}{\Gamma \Rightarrow \Delta}\,\mathrm{Cut}$$

will be denoted as $\mathcal{G}^\infty\mathsf{IL} + \mathrm{Cut}$.

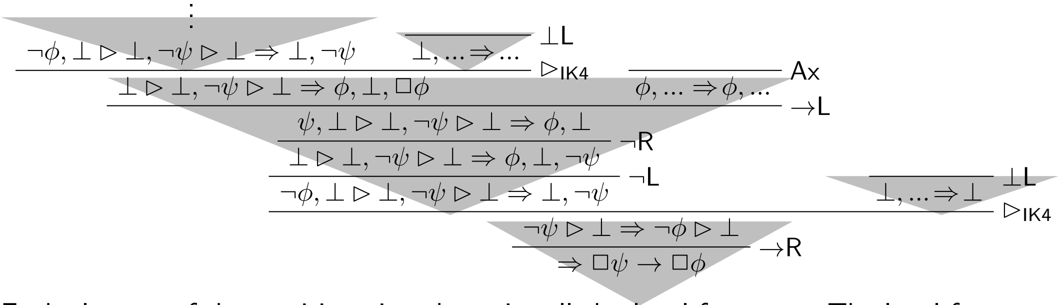
An example of a non-wellfounded proof in $\mathcal{G}^{\infty}$IL

$$
\begin{prooftree}
\AxiomC{$\vdots$}
\noLine
\UnaryInfC{$\neg\phi, \bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \bot, \neg\psi$}
\AxiomC{}
\RightLabel{$\bot$L}
\UnaryInfC{$\bot, \ldots \Rightarrow \ldots$}
\RightLabel{$\rhd_{\mathrm{IK4}}$}
\BinaryInfC{$\bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \phi, \bot, \Box\phi$}
\AxiomC{}
\RightLabel{Ax}
\UnaryInfC{$\phi, \ldots \Rightarrow \phi, \ldots$}
\RightLabel{$\to$L}
\BinaryInfC{$\psi, \bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \phi, \bot$}
\RightLabel{$\neg$R}
\UnaryInfC{$\bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \phi, \bot, \neg\psi$}
\RightLabel{$\neg$L}
\UnaryInfC{$\neg\phi, \bot \rhd \bot, \neg\psi \rhd \bot \Rightarrow \bot, \neg\psi$}
\AxiomC{}
\RightLabel{$\bot$L}
\UnaryInfC{$\bot, \ldots \Rightarrow \bot$}
\RightLabel{$\rhd_{\mathrm{IK4}}$}
\BinaryInfC{$\neg\psi \rhd \bot \Rightarrow \neg\phi \rhd \bot$}
\RightLabel{$\to$R}
\UnaryInfC{$\Rightarrow \Box\psi \to \Box\phi$}
\end{prooftree}
$$

So we have proved that $\mathcal{G}$IL $\vdash \Box\psi \to \Box\phi$, where $\psi = \Box\phi \to \phi$, i.e., axiom L.

Slicing the proof

We define an equivalence relation $\sim$ between the nodes of a proof such that $w \sim v$ if in the shortest path between w and v there is no progress.



Each element of the partition given by $\sim$ is called a local fragment. The local fragment of the root is called the main local fragment and its height is called the local height of the proof. With this structure we can see proofs as corecursive-recursive objects.

Löb's rule

Lemma

Löb's rule

$$\frac{\psi, (\psi, \Gamma) \rhd \bot \Rightarrow \Gamma}{\psi, \Gamma \rhd \bot \Rightarrow \Gamma}\ \textit{Löb}$$

is admissible in $\mathcal{G}\mathsf{IL} + \mathrm{Cut}$.

Proof. Assume we have $\pi \vdash \psi, (\psi, \Gamma) \rhd \bot \Rightarrow \Gamma$. By weakening we obtain a proof $\tau \vdash \psi, (\psi, \Gamma, \bot) \rhd \bot \Gamma, \bot$. We have the following derivation

$$\dfrac{\dfrac{\overset{\tau}{\psi, (\psi, \Gamma, \bot) \rhd \bot \Rightarrow \Gamma, \bot} \quad \dfrac{}{\bot, \ldots \Rightarrow \ldots}\,\bot\mathrm{L} \quad \cdots}{\psi, \Gamma \rhd \bot \Rightarrow \Gamma, \psi \rhd \bot}\,\rhd_{\mathsf{IL}} \quad \overset{\pi}{\psi, (\psi, \Gamma) \rhd \bot \Rightarrow \Gamma}}{\psi, \Gamma \rhd \bot \Rightarrow \Gamma}\,\mathrm{Cut}$$

□

From $\mathcal{G}$IL + Cut to $\mathcal{G}^\infty$IL + Cut

Theorem

Let S be a sequent. If $\mathcal{G}\mathsf{IL} + \mathrm{Cut} \vdash S$, then $\mathcal{G}^\infty\mathsf{IL} + \mathrm{Cut} \vdash S$.

Proof. Via corecursion we define a function α from proofs in $\mathcal{G}$IL + Cut to proofs in $\mathcal{G}^\infty$IL and cases on the shape of the input proof π. The only interesting case is when the last rule of π is $\rhd_{\mathsf{IL}}$. We know that π has shape

$$\dfrac{\left[\begin{array}{c}\pi_i \\ \psi_i, (\psi_i, \Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{array}\right]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \Delta, \psi_m \rhd \phi} \rhd_{\mathsf{IL}} .$$

Then the desired proof is

$$\dfrac{\left[\begin{array}{c}\alpha(\text{Löb}(\pi_i)) \\ \psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi\end{array}\right]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \Delta, \psi_m \rhd \phi} \rhd_{\mathsf{IL}} .$$

$\square$

The set $\mathrm{Sub}(\phi)$

- Let ϕ be a formula. We define the set $\mathrm{Sub}(\phi)$ recursively as follows:

$$\begin{aligned}
&\mathrm{Sub}(p) = \{p\},\\
&\mathrm{Sub}(\bot) = \{\bot\},\\
&\mathrm{Sub}(\phi \to \psi) = \{\phi \to \psi\} \cup \mathrm{Sub}(\phi) \cup \mathrm{Sub}(\psi),\\
&\mathrm{Sub}(\phi \rhd \psi) = \{\phi \rhd \psi, \phi \rhd \bot, \psi \rhd \bot, \bot\} \cup \mathrm{Sub}(\phi) \cup \mathrm{Sub}(\psi).
\end{aligned}$$

- If Γ is a multiset of formulas, $\mathrm{Sub}(\Gamma)$ is the *set*

$$\bigcup \{\mathrm{Sub}(\phi) \mid \phi \in \Gamma\}.$$

- If $S = (\Gamma \Rightarrow \Delta)$ is a sequent, then $\mathrm{Sub}(S)$ is simply the set $\mathrm{Sub}(\Gamma \cup \Delta)$.

$$\frac{[\psi_i, \Phi_{[0,i)} \rhd \bot, \phi \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi]_{m \ldots i \ldots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi_m \rhd \phi, \Delta} \rhd_{\mathrm{IK4}}$$

From $\mathcal{G}^\infty$IL to $\mathcal{G}$IL

Lemma

Let $\pi \vdash S$ in $\mathcal{G}^\infty\mathsf{IL}$ and ϕ be a formula occurring in π. Then $\phi \in \mathrm{Sub}(S)$.

Proof. By the induction on the length of the node where ϕ appears. □

Theorem

For any finite set Λ of formulas, we have that

$$\mathcal{G}^\infty\mathsf{IL} \vdash \Gamma \Rightarrow \Delta \text{ implies } \mathcal{G}\mathsf{IL} \vdash \Lambda \rhd \bot, \Gamma \Rightarrow \Delta.$$

Proof. Let $\pi \vdash \Gamma \Rightarrow \Delta$ in $\mathcal{G}^\infty$IL. By induction on the lexicographical order

$$\Big(|\mathrm{Sub}(\Gamma \Rightarrow \Delta) \setminus \Lambda|, \mathrm{lhg}(\pi)\Big)$$

and cases on the last rule of π. □

Admissibility and eliminability

Let $\mathcal{G}$ be any kind (wellfounded or non-wellfounded) of sequent calculus and R be a rule. We say that

1. R is *admissible* in $\mathcal{G}$ if for any instance $(S_0, \ldots, S_{n-1}, S)$ of R we have

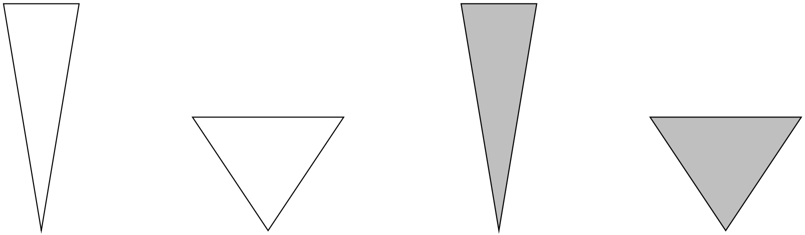
$$\mathcal{G} \vdash S_0, \ldots, \mathcal{G} \vdash S_{n-1} \text{ implies } \mathcal{G} \vdash S.$$

2. R is *eliminable* in $\mathcal{G}$ if $\mathcal{G} + \mathrm{Cut}$ proves the same sequents as $\mathcal{G}$.

Note that eliminability implies admissibility. However, if $\mathcal{G}$ is a wellfounded sequent calculus we also have the inverse direction, but not in the non-wellfounded case.

Coalgebraic Proof Translations of Non-Wellfounded Proofs

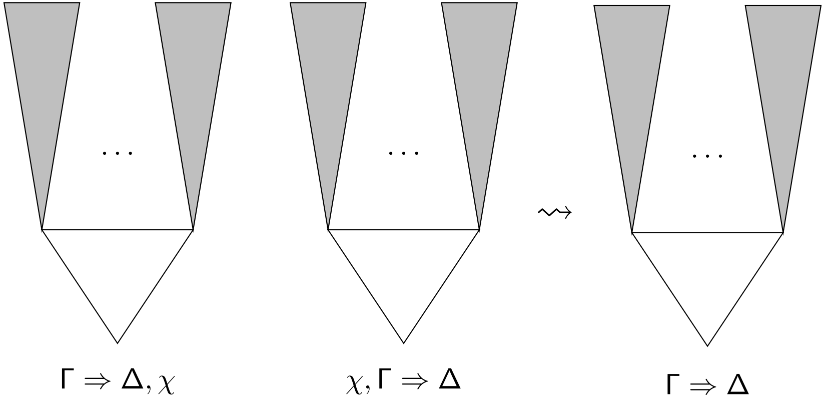
Borja Sierra Miranda, Thomas Studer and Lukas Zenger, "Coalgebraic Proof Translations of Non-Wellfounded Proofs". In Agata Ciabattoni, David Gabelaia and Igor Sedlár (eds). (2024) Advances in Modal Logic, Vol. 15. College Publications



On the left: non-wellfounded proof and local fragment without cuts;
on the right: non-wellfounded proof and local fragment that can contain cuts

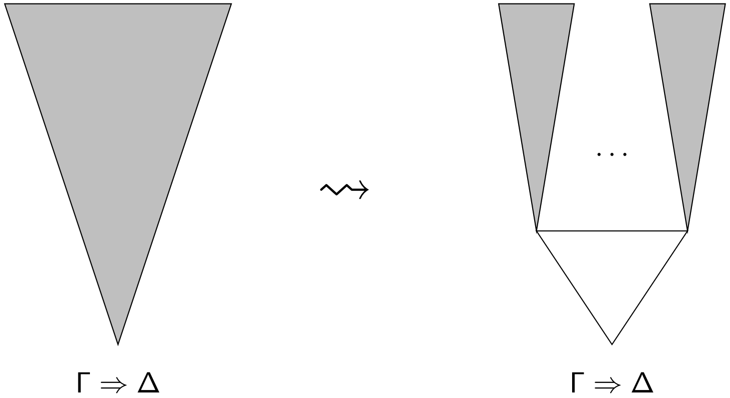
Coalgebraic Proof Translations of Non-Wellfounded Proofs

(1) local admissibility



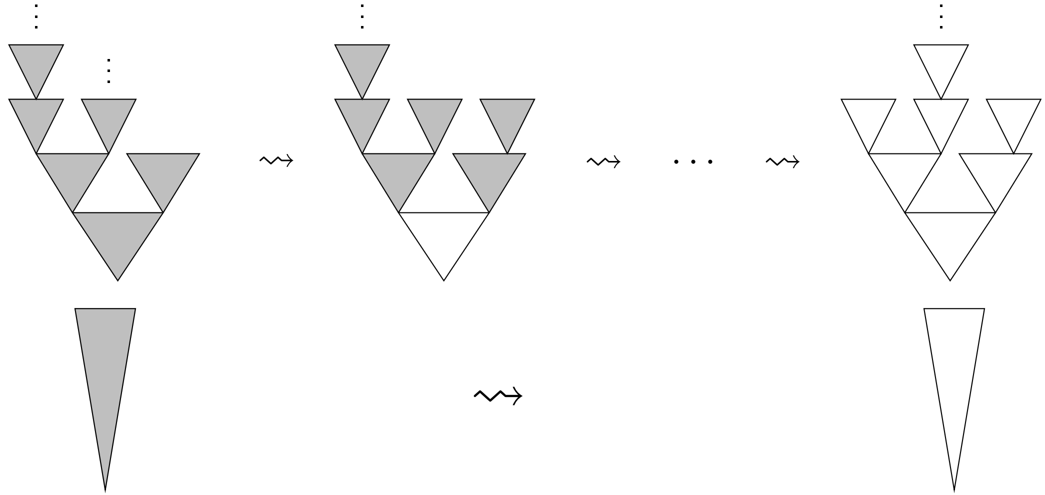
Coalgebraic Proof Translations of Non-Wellfounded Proofs

(2) step (local eliminability)



Coalgebraic Proof Translations of Non-Wellfounded Proofs

(3) applying step (eliminability)



Local cut admissibility

If π is a non-wellfounded proof, we denote by $\mathrm{lhg}(\pi)$ its local height.

Lemma

Assume we have proofs $\pi \vdash \Gamma \Rightarrow \Delta, \chi$ and $\tau \vdash \chi, \Gamma \Rightarrow \Delta$ in $\mathcal{G}^\infty\mathsf{IL} + \mathrm{Cut}$ which are locally cut-free. Then there is $\rho \vdash \Gamma \Rightarrow \Delta$ in $\mathcal{G}^\infty\mathsf{IL} + \mathrm{Cut}$ which is locally cut-free.

Proof. By induction on the lexicographic order of the pairs $(|\chi|, \mathrm{lhg}(\pi) + \mathrm{lhg}(\tau))$ and cases on the shape of the cut. Let us see the modal case. Assume we have proofs

$$\dfrac{\left[\overset{\pi_i}{\psi_i, (\Phi_{[0,i)}, \phi) \rhd \bot \Rightarrow \Phi_{[0,i)}, \phi}\right]_{m \dots i \dots 0}}{\{\phi_i \rhd \psi_i\}_{i<m}, \{\phi'_j \rhd \psi'_j\}_{j<n, j \neq k}, \Gamma \Rightarrow \psi_m \rhd \phi, \psi'_n \rhd \phi', \Delta,} \rhd_{\mathrm{IK4}}$$

$$\dfrac{\left[\overset{\tau_i}{\psi'_j, (\Phi'_{[0,j)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,j)}, \phi'}\right]_{n \dots j \dots 0}}{\{\phi'_j \rhd \psi'_j\}_{j<n}, \{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi'_n \rhd \phi', \Delta} \rhd_{\mathrm{IK4}},$$

where $\psi_m \rhd \phi = \phi'_k \rhd \psi'_k$.

We are going to define a proof

$$\dfrac{\rho'_n \quad \cdots \quad \rho'_{k+1} \quad \rho_{m-1} \quad \cdots \quad \rho_0 \quad \rho'_{k-1} \quad \cdots \quad \rho'_0}{\{\phi'_j \rhd \psi'_j\}_{j<n, j\neq k}, \{\phi_i \rhd \psi_i\}_{i<m}, \Gamma \Rightarrow \psi'_n \rhd \phi', \Delta} \rhd_{\mathsf{IK4}}$$

where $\rhd_{\mathsf{IK4}}$ has been applied with order

$$\phi'_0 \rhd \psi'_0, \ldots, \phi'_{k-1} \rhd \psi'_{k-1}, \phi_0 \rhd \psi_0, \ldots, \phi_{m-1} \rhd \psi_{m-1}, \phi'_{k+1} \rhd \psi'_{k+1}, \ldots, \phi'_{n-1} \rhd \psi'_{n-1}$$

and principal formula $\psi'_n \rhd \phi'$. We need to define ρ'_js and ρ_is such that

$$\rho'_j \vdash \psi'_j, (\Phi'_{[0,k)\cup(k,j)}, \Phi_{[0,m)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,k)\cup(k,j)}, \Phi_{[0,m)}, \phi', \qquad \text{for } j \in (k, n];$$

$$\rho_i \vdash \psi_i, (\Phi'_{[0,k)}, \Phi_{[0,i)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,k)}, \Phi_{[0,i)}, \phi', \qquad \text{for } i \in [0, m);$$

$$\rho'_j \vdash \psi'_j, (\Phi'_{[0,j)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,j)}, \phi', \qquad \text{for } j \in [0, k).$$

Then we can define $\rho'_j = \tau_j$ for $j \in [0, k)$, let us see the other definitions.

We notice that the necessitation rule

$$\dfrac{\phi, \Sigma \rhd \bot \Rightarrow \Sigma}{\Sigma \rhd \bot, \Gamma \Rightarrow \phi \rhd \bot, \Delta}\ \mathrm{Nec}$$

is admissible in $\mathcal{G}^\infty\mathsf{IL}(+\mathrm{Cut})$. Let us define ρ_i, first let $\Phi := \Phi'_{[0,k)}, \Phi_{[0,i)}$.

$$\dfrac{\dfrac{\dfrac{\begin{matrix}\tau_k\\ \psi'_k, (\Phi'_{[0,k)}, \phi') \rhd \bot \Rightarrow \Phi'_{[0,k)}, \phi'\end{matrix}}{\psi_i, (\Phi, \phi') \rhd \bot \Rightarrow \Phi, \psi'_k, \phi', \psi'_k \rhd \bot}\ \mathrm{N} \quad \dfrac{\dfrac{\pi_i}{\psi_i, (\Phi, \phi, \phi') \rhd \bot \Rightarrow \Phi, \phi, \phi'}\ \mathrm{W}}{\psi_i, (\Phi, \psi'_k, \phi') \rhd \bot \Rightarrow \Phi, \psi'_k, \phi'}}{\psi_i, (\Phi, \phi') \rhd \bot \Rightarrow \Phi, \psi'_k, \phi'}\ \mathrm{C} \quad \dfrac{\tau_k}{\psi_i, \psi'_k, (\Phi, \phi') \rhd \bot \Rightarrow \Phi, \phi'}\ \mathrm{W}}{\psi_i, (\Phi, \phi') \rhd \bot \Rightarrow \Phi, \phi'}\ \mathrm{C}$$

The definition of ρ'_j for $j \in (k, n]$ is similar but more involved

$$\dfrac{\dfrac{\dfrac{\tau_k}{\cdots}\ \mathrm{N} \quad \dfrac{\dfrac{\dfrac{\pi_m}{\cdots}\ \mathrm{N} \quad \dfrac{\tau_j}{\cdots}\ \mathrm{W}}{\cdots}\ \mathrm{C} \quad \dfrac{\pi_m}{\cdots}\ \mathrm{W}}{\cdots}\ \mathrm{C}}{\cdots}\ \mathrm{C} \quad \dfrac{\tau_k}{\cdots}\ \mathrm{W}}{\cdots}\ \mathrm{C}$$

$\square$

Cut elimination for $\mathcal{G}^\infty$IL

By just applying the previous lemma corecursively to local proof fragments we get the desired result about the cut elimination for $\mathcal{G}^\infty$IL.

Theorem

If $\mathcal{G}^\infty\mathsf{IL} + \mathrm{Cut} \vdash S$, *then* $\mathcal{G}^\infty\mathsf{IL} \vdash S$.

And as a corollary of all our previous results, we can obtain cut elimination in the original system.

Corollary

If $\mathcal{G}\mathsf{IL} + \mathrm{Cut} \vdash S$, *then* $\mathcal{G}\mathsf{IL} \vdash S$.

Conclusions and Future Work

Conclusions:

1. The "amount of non-wellfoundedness" needed to treat provability logics is quite low. In addition, it can be used to provide new results.
2. A system for ILP is easy to derive from the system of ILP.
3. Also a system for il (unary interpretability logic) is easy to derive.

Future work:

1. Exploit the absence of diagonal formulas to provide Lyndon uniform interpolation.
2. Uniform interpolation in provability logics with bad repetitions e.g. Grz.
3. Extensions of IL and il.
4. Using the uniformity of the methodology, provide an abstract theory of "cyclic/non-wellfounded companions".

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