Formal Verification using Parity Games

Mathias N. Justesen

DTU Compute, Technical University of Denmark (DTU)

Background

- Many problems within formal verification can be reduced to solving parity games
  - Model checking (Stirling, 1995)
  - Controller synthesis (Arnold *et al.* , 2003)
  - Satisfiability (Friedmann & Lange, 2009b)

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  - PBES to parity game

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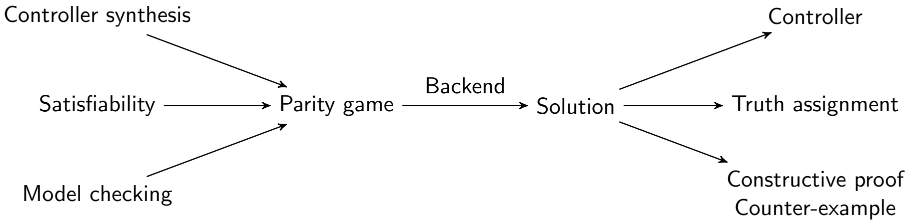
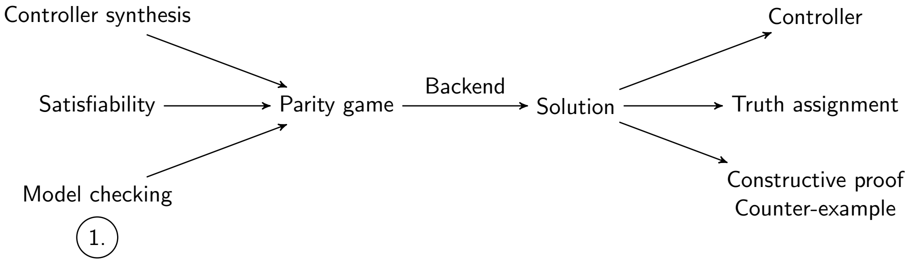
  - mCRL2 and LTSmin
  - PBES to parity game
- Verification framework based on parity game solving

Framework


Controller synthesis
Satisfiability
Model checking
Parity game
Backend
Solution
Controller
Truth assignment
Constructive proof
Counter-example

Framework



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   - Semantics based on evaluation games
   - Conversion from evaluation game to parity game

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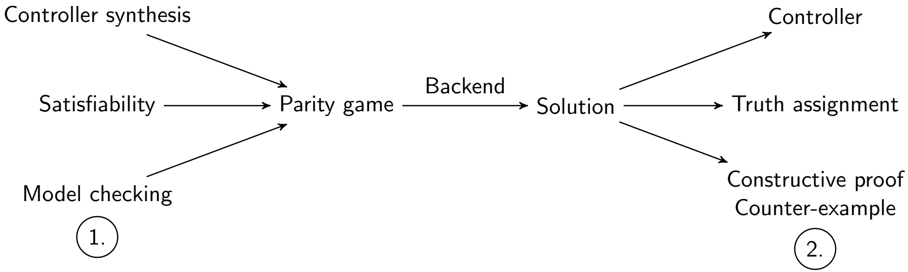

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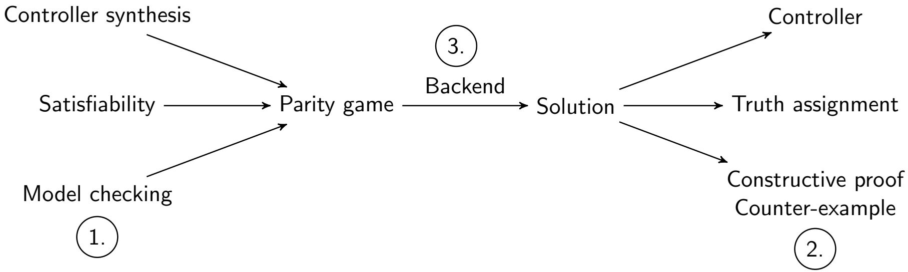

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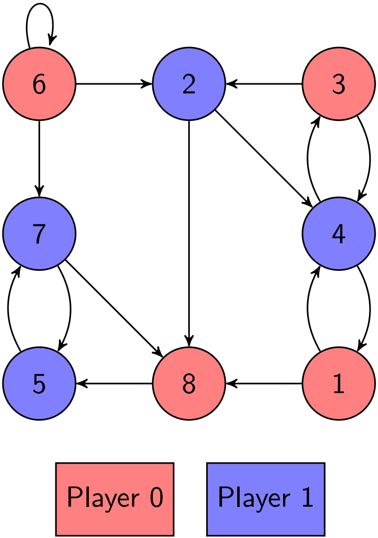
   - Semantics based on evaluation games
   - Conversion from evaluation game to parity game
2. Use solution to construct proof or counter-example
3. Backend based on PGSolver
   - Solve parity games in normal form

Parity Game



Model Checking

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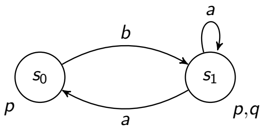

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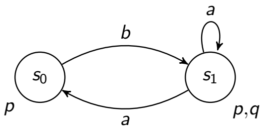
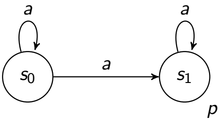


- Formulas of modal μ-calculus given proposition variables P and actions A:

$$\varphi ::= \top \mid \bot \mid p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \langle a \rangle \varphi \mid [a]\varphi \mid \mu x.\varphi \mid \nu x.\varphi$$

  where $p, x \in P$ and $a \in A$

Evaluation Game



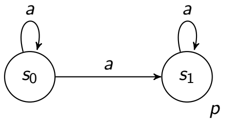
$\mu x.p \vee [a]x$

Player 0: Prove

Player 1: Disprove

Construction cf. (Venema, 2008)

Evaluation Game



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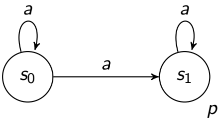

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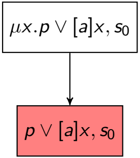

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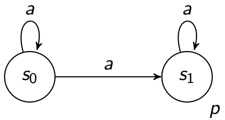

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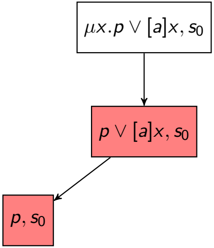

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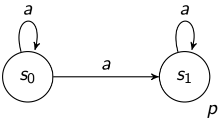

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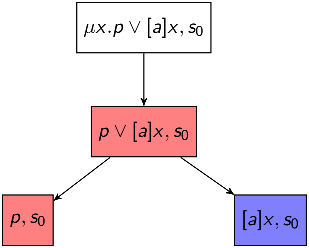

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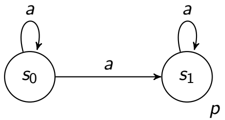

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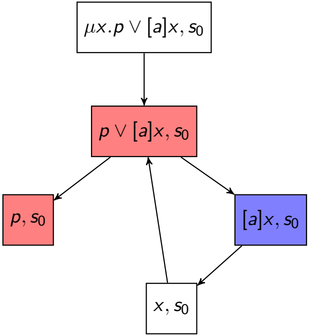

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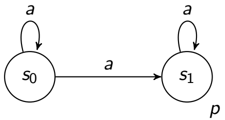

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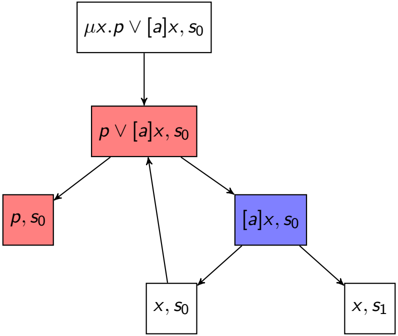

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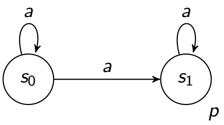

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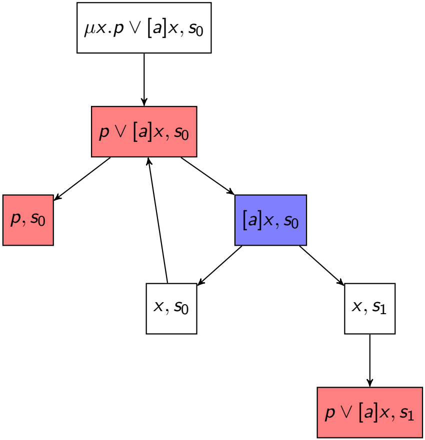

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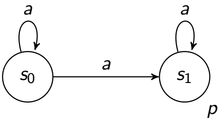

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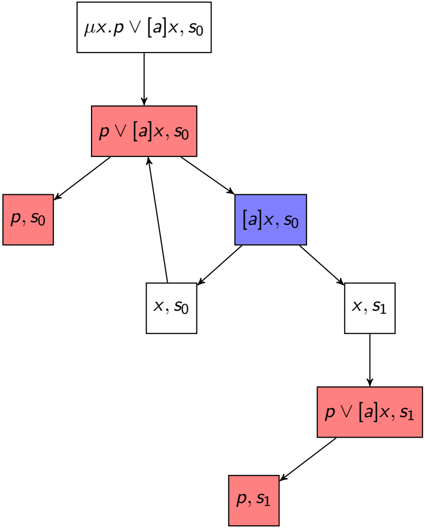

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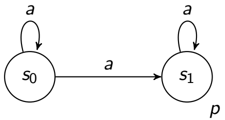

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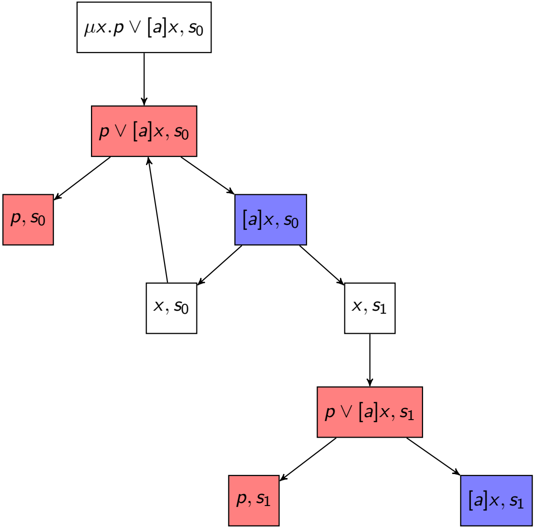

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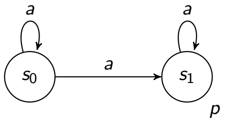

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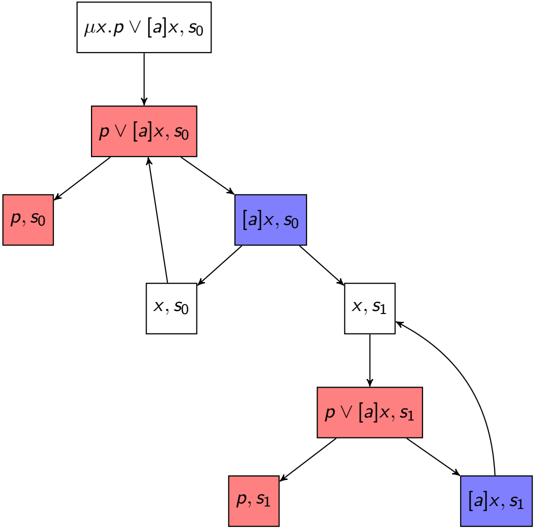

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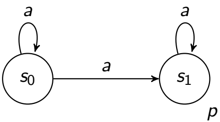

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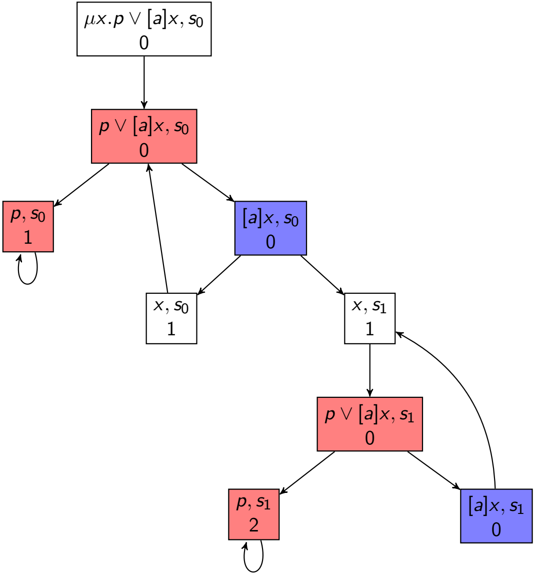

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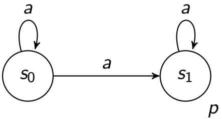

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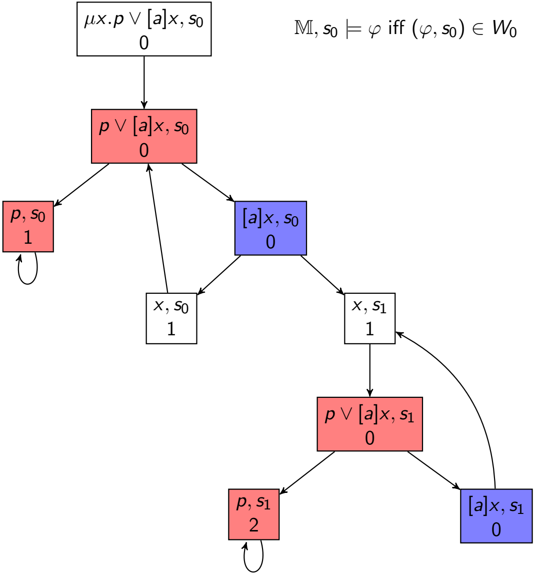

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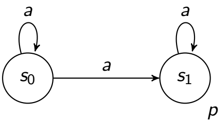

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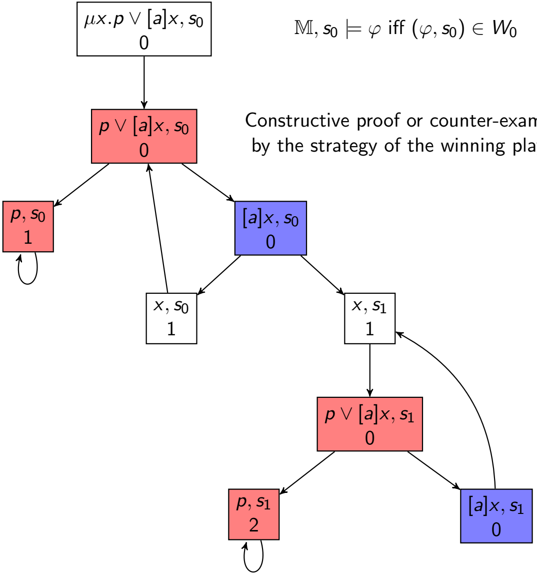

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Constructive proof or counter-example by the strategy of the winning player

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- Normal-Form Algorithm 2
  - Improved version of Normal-Form Algorithm 1

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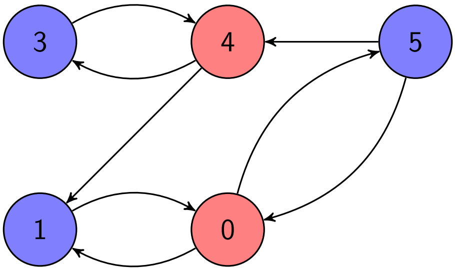
  - Player 0 controls only nodes of even priority, and
  - Player 1 controls only nodes of odd priority

Example



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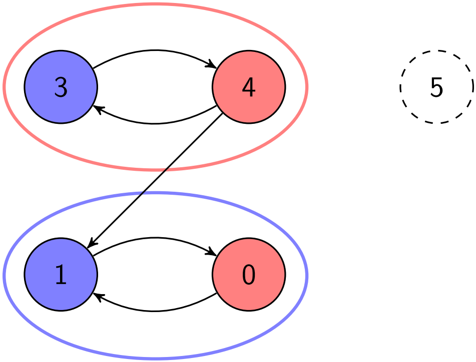

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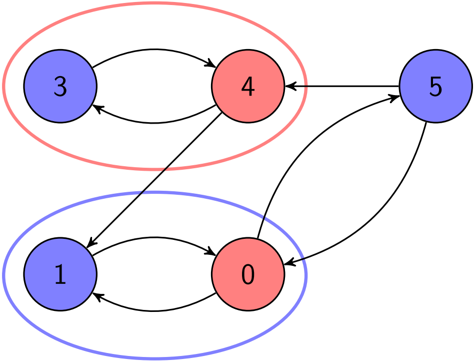

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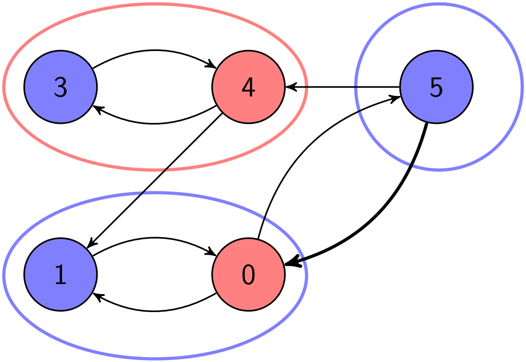

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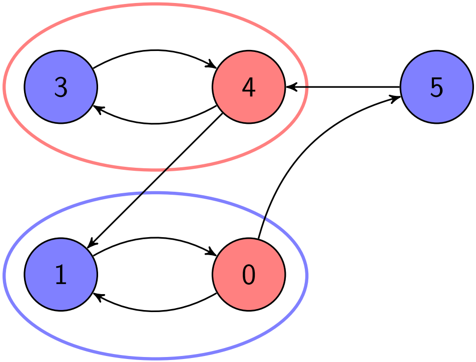

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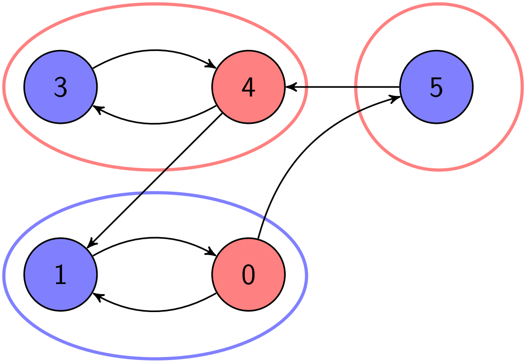

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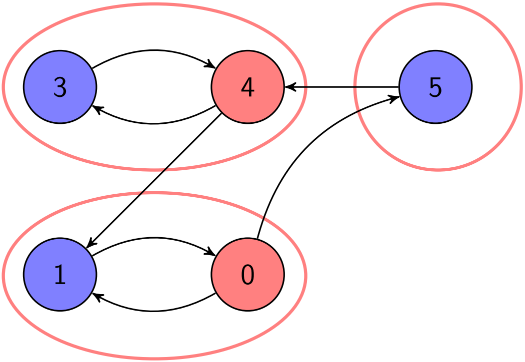

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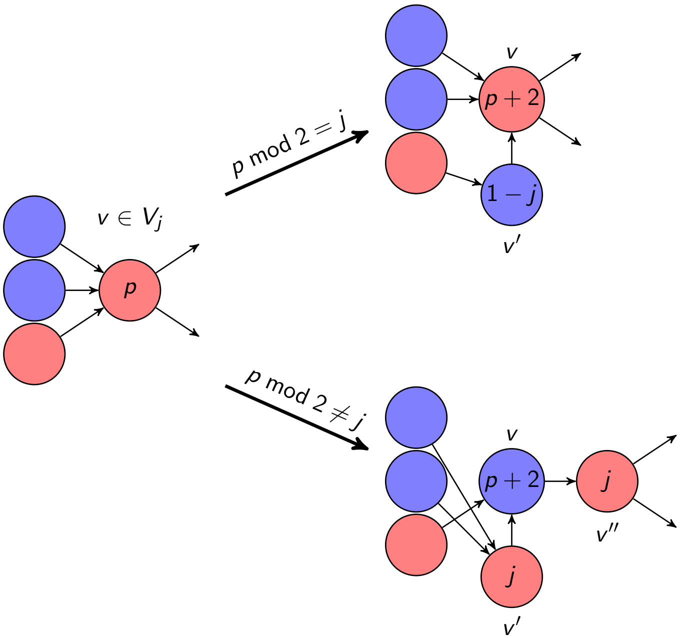
- Algorithms restricted to games in normal form

Transformation



Comparison of Algorithms

$n, d, deg_{min}, deg_{max}$	Not NF			Pre-NF			NF		
	Zie	NF1	NF2	Zie	NF1	NF2	Zie	NF1	NF2
100, 100, 2, 4	0.00	10.55	0.42	0.00	10.58	0.41	0.00	0.04	0.02
100, 100, 2, 10	0.00	6.13	0.29	0.00	6.16	0.28	0.00	0.01	0.01
100, 100, 2, 100	0.00	3.47	0.18	0.00	3.45	0.19	**0.01**	**0.01**	**0.01**
200, 200, 2, 4	0.00		11.01	0.00		10.78	0.01	0.43	0.23
200, 200, 2, 10	0.00		2.37	0.00		2.29	0.01	0.22	0.16
200, 200, 2, 200	0.01	69.29	2.29	0.01	52.05	2.27	**0.05**	**0.05**	**0.03**
500, 500, 2, 4	0.00			0.01			0.07		
500, 500, 2, 10	0.01			0.03			0.10	13.24	6.31
500, 500, 2, 500	0.07		78.01	0.08		77.18	**1.11**	**1.04**	**0.73**
Rec. ladder 5	0.00	0.03	0.01						
Rec. ladder 10	0.01	5.94	0.75						
Rec. ladder 15	0.07		94.36						

Testing the Limits

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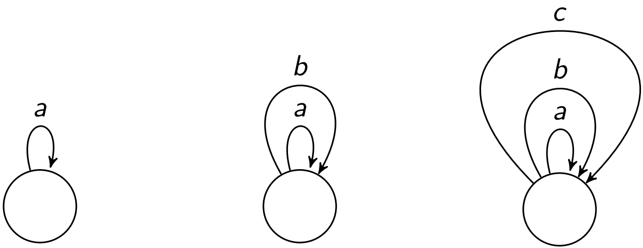

(a) $\mathbb{L}_1$ (b) $\mathbb{L}_2$ (c) $\mathbb{L}_3$

Testing the Limits

LTS	Nodes	n	Time
$\mathbb{L}_1$	12.000	1024	3:27.4
$\mathbb{L}_2$	786.000	16	0:03.6
$\mathbb{L}_2$	1.573.000	17	0:03.8
$\mathbb{L}_3$	413.000	10	0:01.8
$\mathbb{L}_3$	1.240.000	11	0:05.6
$\mathbb{L}_3$	3.720.000	12	0:07.6

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State space: $O\big(|\mathbb{M}| \cdot |Sfor(\varphi)|\big)$

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  - Symbolic representation of parity games (Kant & van de Pol, 2014)

References I

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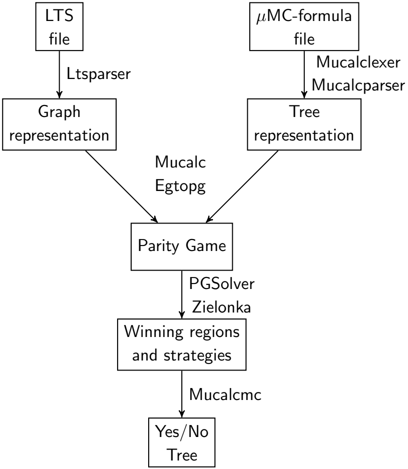
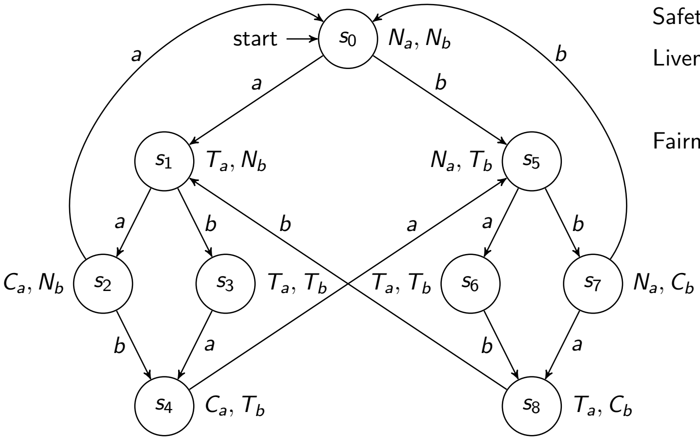
Implementation



Mutual Exclusion



Safety: $\Box\neg(C_a \wedge C_b)$

Liveness: $\Diamond C_a$

$\Box(T_a \rightarrow \Diamond C_a)$

Fairness: $\Box\Diamond C_a$

$\Box\Diamond T_a \rightarrow \Box\Diamond C_a$

Example from (Artale, 2011)