Acyclic attribute evaluation in a dependently typed setting

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Example: Semantics of binary numbers

$$
\begin{aligned}
\mathtt{N} &\rightarrow \mathtt{L} \\
\mathtt{L}_1 &\rightarrow \mathtt{L}_2\mathtt{B}
\end{aligned}
$$

$$
\begin{aligned}
\mathtt{L} &\rightarrow \mathtt{B} \\
\mathtt{B} &\rightarrow \mathtt{0} \\
\mathtt{B} &\rightarrow \mathtt{1}
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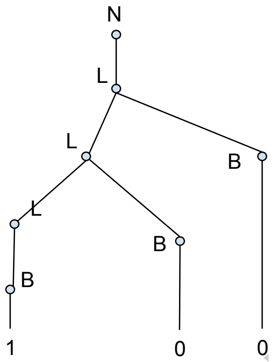

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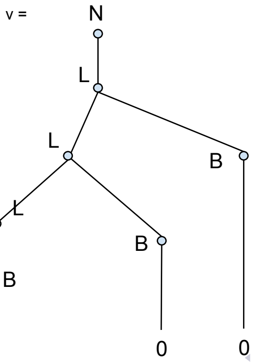

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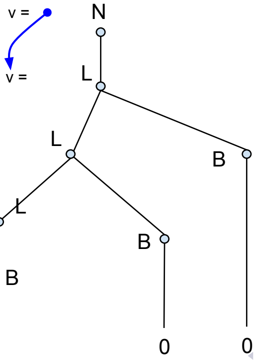

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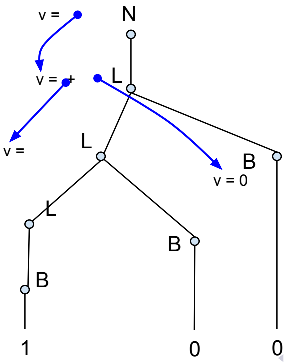

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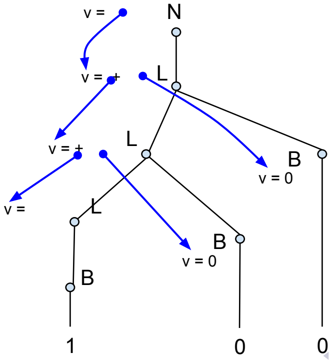

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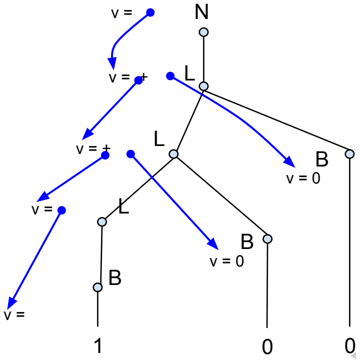

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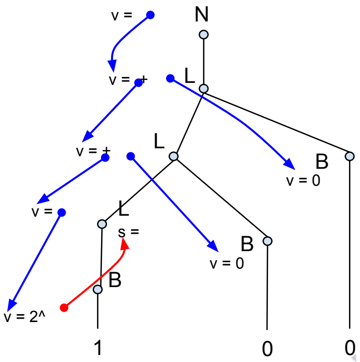

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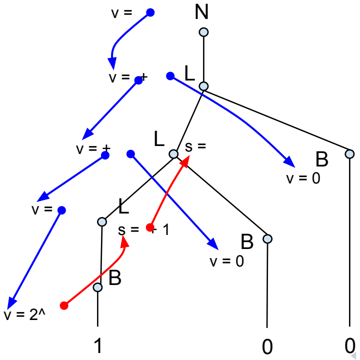

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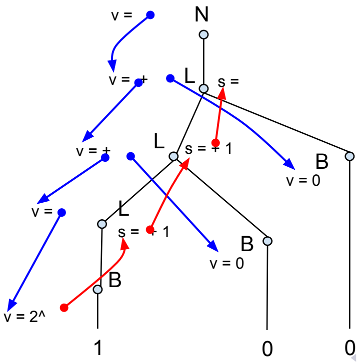

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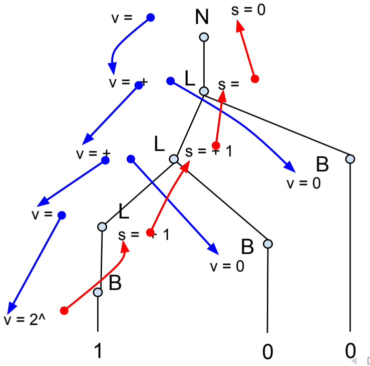

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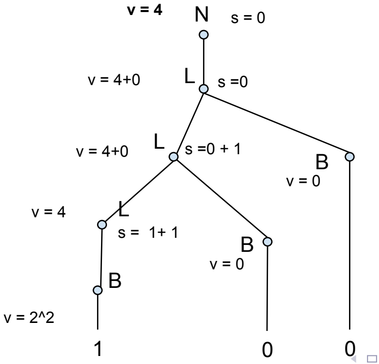

Basic definitions: Parse trees

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Attr : Set
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Rule : N → List N → Set
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mutual
  Forest = List (∃(X : N), Tree X)

  data Tree : N → Set where
    node : ∀ {X} → (f : Forest)
         → Rule X (map proj₁ f) → Tree X
```

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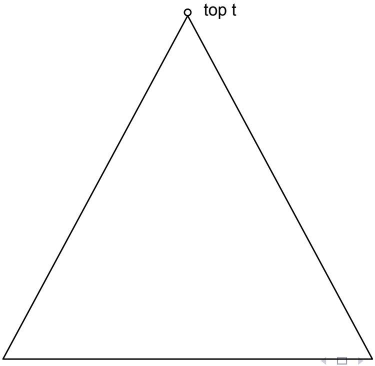

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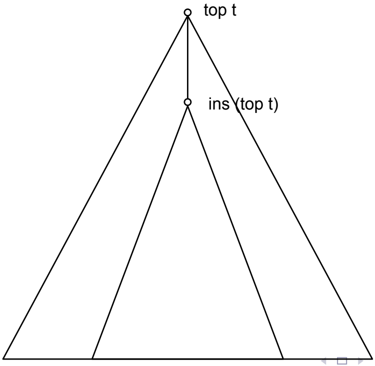

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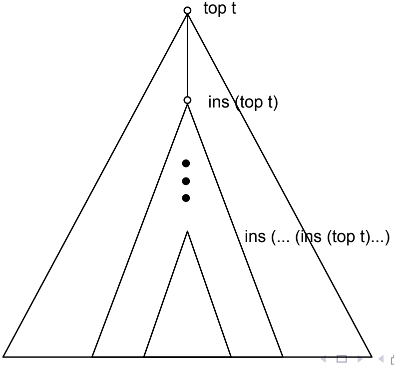

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    b|[ p , a₁ ] ⇓ [ ins p , a₂ ]
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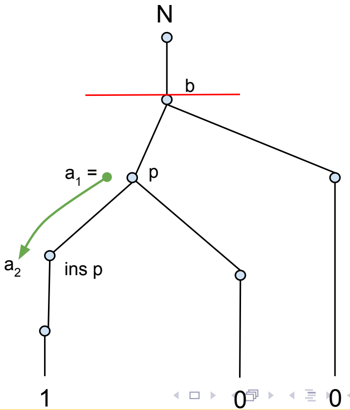

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$$\mathtt{data}\ \_|[\_]\overset{+/*}{\rightsquigarrow}[\_]\ \{\mathtt{Z} : \mathtt{N}\}\{\mathtt{t} : \mathtt{Tree}\ \mathtt{Z}\}(\mathtt{b} : \mathtt{PosTree}\ \mathtt{t}) : \forall\{\mathtt{X}\ \mathtt{Y}\ \mathtt{t}_1\ \mathtt{t}_2\} \to \mathtt{PosTree}\ \mathtt{t}_1 \times \mathtt{Attr} \to \mathtt{PosTree}\ \mathtt{t}_2 \times \mathtt{Attr} \to \mathtt{Set}$$

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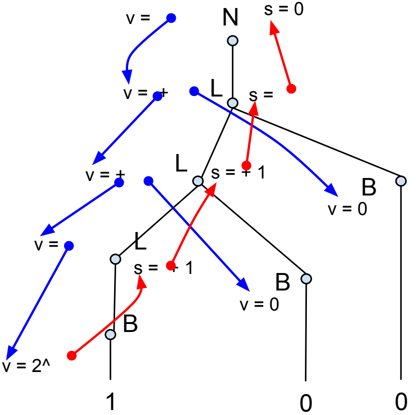
$$\texttt{q |[q , v]} \overset{+}{\rightsquigarrow} \texttt{[q , s]}$$



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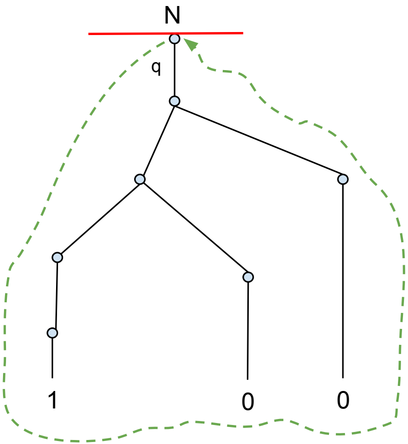
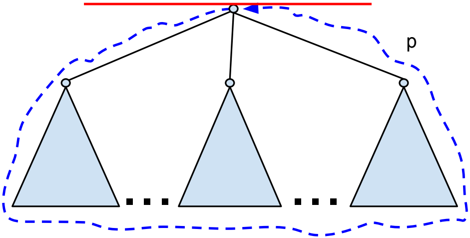


Cycle decomposition



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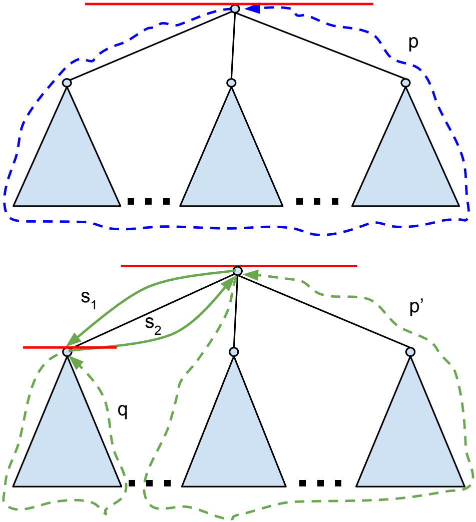

p ≡ s_1 ++ q ++ s_2 ++ p'

Induction principle

Theorem (Induction principle)

Let `P` *be a property of paths of type* `p |[ p ] f [ p ]`, *where* `p` *is any position in any tree. Then to conclude that* `P` *holds for all cycles on all trees, we need to establish the following:*

- `P` *holds for empty cycles on all trees.*
- *Any proofs that* `P` *holds for a cycle* `q` *of type* `(ins p c)|[ ins p c ] f [ ins p c ]` *and* `P` *holds for a cycle* `p'` *of type* `p |[ p ] f [ p ]` *can be converted into a proof that* `P` *holds for the cycle* $\mathtt{s}_1$ `++ q ++` $\mathtt{s}_2$ `++ p'` *where* $\mathtt{s}_1$ *and* $\mathtt{s}_2$ *are steps down from and up to* `p`.

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dependencies {X} t = ...  -- reachability on graph
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_≈_ : {X : N} → Tree X → Tree X → Set
t ≈ t' = dependencies t ≡ dependencies t'
```
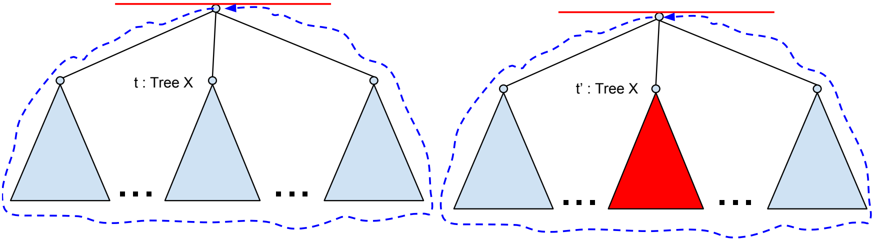

Substitution lemma

If t $\approx$ t' then `node (f₁ ++ [ t ] ++ f₂)` $\approx$ `node (f₁ ++ [ t' ] ++ f₂))`

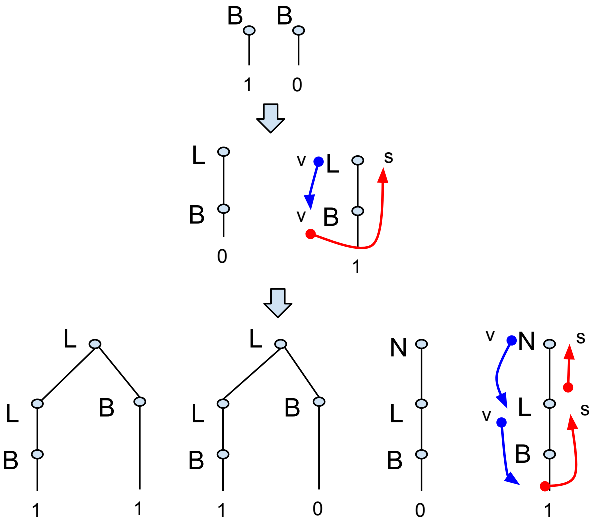


Checking for circularity: Algorithm

`treesToCheck : Rules → Forest`

1. Initialize accumulator with terminal rules
2. Build new trees from existing ones in accumulator
3. For each new tree `t`:
    1. Compute the `dependencies t`
    2. If there is no tree `t'` in accumulator such that `t` $\approx$ `t'` then add `t` to accumulator
4. If at least one tree was added then go to step 2, otherwise return accumulator.

Checking for circularity: Example



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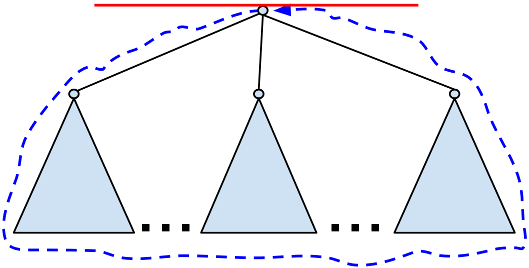

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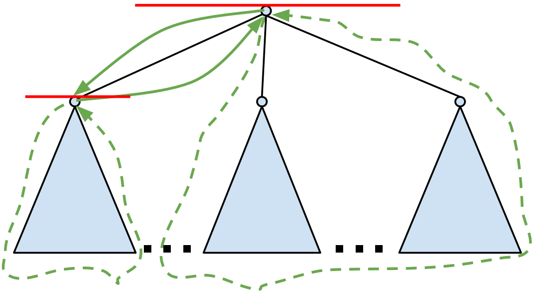

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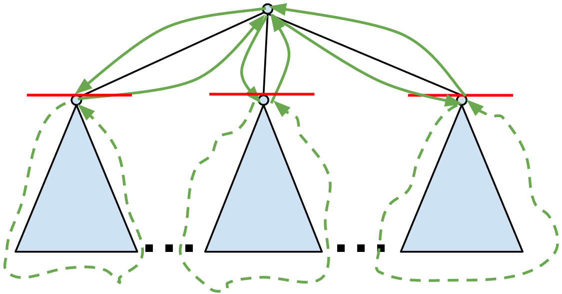

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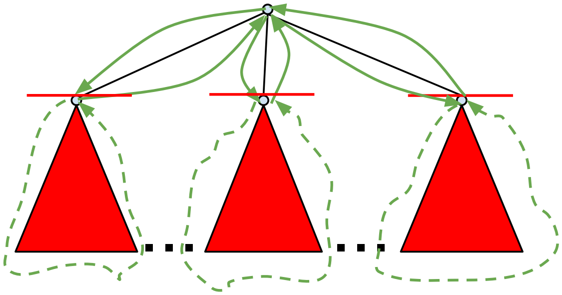

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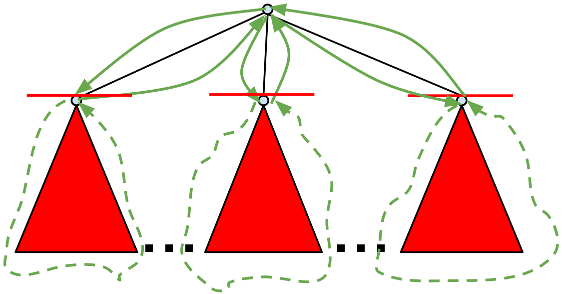

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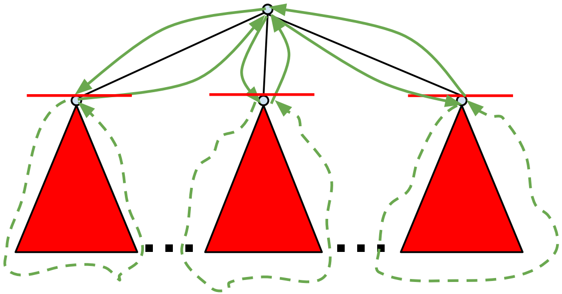

```
circular? : Rules → Bool
circular? = ⋁ { a ≟ b | t ← treesToCheck Rs, (a, b) ← dependencies t }
```

Thank you for your attention!
Questions?