Towards Normalization by Evaluation for Lambek Calculus

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The syntactic calculus $\mathbf{L}$ of Lambek [8] is a deductive system which is primarily employed in mathematical studies of sentence structure in natural language. From a logical perspective, it provides a proof system for the multiplicative fragment of intuitionistic noncommutative linear logic [2, 12], comprising only of the tensor product $\otimes$ and the two ordered implications $\diagup$ and $\diagdown$ as connectives.

The metatheory of the Lambek calculus has been thoroughly developed in the past decades, in particular its categorical semantics by Lambek himself [9, 10]. The Lambek calculus enjoys cut elimination [8] and various normalization procedures, e.g. by Hepple for the tensor-free fragment [7] or more recently by Amblard and Retoré [5], aimed at the reduction of the proof-search space and consequently the number of possible derivation of a given sequent. Various diagrammatic calculi and proof nets for the Lambek calculus have also been proposed [14, 11].

In this work, we discuss the natural deduction presentaton of the Lambek calculus $\mathbf{L}$, together with a calculus $\mathbf{L}_{\beta\eta}$ consisting of $\beta\eta$-long normal forms, i.e. derivations that do not contain any redexes, and no further η-expansion is applicable. The calculus $\mathbf{L}_{\beta\eta}$ is a particular fragment of the calculus of normal forms for intuitionistic noncommutative linear logic introduced by Polakov and Pfenning [12]. The normalization algorithm, sending each derivation in $\mathbf{L}$ to its $\beta\eta$-long normal form in $\mathbf{L}_{\beta\eta}$, is an instance of *normalization by evaluation* [6, 3], where the effective normalization procedure factors through a denotational Kripke model of the syntactic calculus. A similar procedure, for the simpler case of the Lambek calculus without tensor product, can be extrapolated from Polakov's PhD thesis [13].

Lambek Calculus in Natural Deduction

The Lambek calculus $\mathbf{L}$ has formulae given by the grammar: $A, B ::= p \mid A \otimes B \mid B \diagup A \mid A \diagdown B$, where p comes from a given set At of atomic formulae, $\otimes$ is a multiplicative conjunction (a.k.a. tensor product), and $\diagup$ and $\diagdown$ are left and right implications (a.k.a. left and right residuals, or internal homs). Sequents in $\mathbf{L}$ are pairs $\Gamma \vdash A$, where Γ is an ordered (possibly empty) list of formulae, called context, and A is a single formula. Derivations in $\mathbf{L}$ are generated by the following inference rules:

$$\frac{}{A \vdash A}\ \mathsf{ax} \qquad \frac{\Gamma \vdash A \quad \Delta \vdash B}{\Gamma, \Delta \vdash A \otimes B}\ I_\otimes \qquad \frac{\Gamma \vdash A \otimes B \quad \Delta_0, A, B, \Delta_1 \vdash C}{\Delta_0, \Gamma, \Delta_1 \vdash C}\ E_\otimes$$

$$\frac{\Gamma, A \vdash B}{\Gamma \vdash B \diagup A}\ I_\diagup \qquad \frac{A, \Gamma \vdash B}{\Gamma \vdash A \diagdown B}\ I_\diagdown \qquad \frac{\Gamma \vdash B \diagup A \quad \Delta \vdash A}{\Gamma, \Delta \vdash B}\ E_\diagup \qquad \frac{\Gamma \vdash A \quad \Delta \vdash A \diagdown B}{\Gamma, \Delta \vdash B}\ E_\diagdown$$

We write $f : \Gamma \vdash A$ to indicate that f is a particular derivation of $\Gamma \vdash A$. We call Fma and Cxt the sets of formulae and contexts, respectively.

Substitution, i.e. the cut rule, is admissible in $\mathbf{L}$ [8].

$$\frac{\Gamma \vdash A \quad \Delta_0, A, \Delta_1 \vdash C}{\Delta_0, \Gamma, \Delta_1 \vdash C}\ \mathsf{cut}$$

Derivations in $\mathbf{L}$ can be identified modulo a certain $\beta\eta$-equivalence relation $\sim$, which we omit from this extended abstract.

Canonical representatives of equivalence classes for the congruence $\sim$, i.e. $\beta\eta$-long normal forms, can be organized in another calculus $\mathbf{L}_{\beta\eta}$. Sequents in $\mathbf{L}_{\beta\eta}$ have two shapes, $\Gamma \Uparrow A$ and $\Gamma \Downarrow A$. In the literature on intuitionistic propositional logic (i.e. simply typed λ-calculus, via Curry-Howard correspondence), derivations of $\Gamma \Uparrow A$ are called *normal forms*, while derivations of $\Gamma \Downarrow A$ are called *neutrals*. Derivations in $\mathbf{L}_{\beta\eta}$ are generated by the following inference rules:

$$\frac{}{A \Downarrow A}\ \mathsf{ax} \qquad \frac{\Gamma \Uparrow A \quad \Delta \Uparrow B}{\Gamma, \Delta \Uparrow A \otimes B}\ I_{\otimes} \qquad \frac{\Gamma \Downarrow A \otimes B \quad \Delta_0, A, B, \Delta_1 \Uparrow C}{\Delta_0, \Gamma, \Delta_1 \Uparrow C}\ E_{\otimes} \qquad \frac{\Gamma \Downarrow p}{\Gamma \Uparrow p}\ \mathsf{sw}_{\Downarrow}$$

$$\frac{\Gamma, A \Uparrow B}{\Gamma \Uparrow B \diagup A}\ I_{\diagup} \qquad \frac{A, \Gamma \Uparrow B}{\Gamma \Uparrow A \diagdown B}\ I_{\diagdown} \qquad \frac{\Gamma \Downarrow B \diagup A \quad \Delta \Uparrow A}{\Gamma, \Delta \Downarrow B}\ E_{\diagup} \qquad \frac{\Gamma \Uparrow A \quad \Delta \Downarrow A \diagdown B}{\Gamma, \Delta \Downarrow B}\ E_{\diagdown}$$

Soundness of $\mathbf{L}_{\beta\eta}$ wrt. $\mathbf{L}$ is evident: each $\mathbf{L}_{\beta\eta}$-derivation can be embedded into $\mathbf{L}$ via functions $\mathsf{emb}_{\Uparrow} : \Gamma \Uparrow A \to \Gamma \vdash A$ and $\mathsf{emb}_{\Downarrow} : \Gamma \Downarrow A \to \Gamma \vdash A$, which simply change $\Uparrow$ and $\Downarrow$ to $\vdash$ and erase all applications of the rule $\mathsf{sw}_{\Downarrow}$.

Normalization by Evaluation

$\mathbf{L}_{\beta\eta}$ is also complete wrt. $\mathbf{L}$. The proof of completeness corresponds to the construction of a normalization algorithm nbe, taking a derivation of $\Gamma \vdash A$ and returning a derivation of $\Gamma \Uparrow A$, satisfying the two following properties, for all derivatons $t, u : \Gamma \vdash A$: (i) $t \sim u \to \mathsf{nbe}\ t = \mathsf{nbe}\ u$; (ii) $t \sim \mathsf{emb}_{\Uparrow}\ (\mathsf{nbe}\ t)$. The procedure nbe is defined following the normalization by evaluation (NbE) methodology: (*i*) Describe a Kripke model of $\mathbf{L}$ and its equational theory $\sim$. This provides the existence of an element $[\![t]\!]$ in the model, for each derivation t in $\mathbf{L}$, such that $[\![t]\!] = [\![u]\!]$ whenever $t \sim u$; (*ii*) Construct a reification function sending an element of the Kripke model to a normal form in $\mathbf{L}_{\beta\eta}$, so that $\mathsf{nbe}\ t$ is defined as the reification of $[\![t]\!]$.

The Kripke Model. It is the presheaf category $\mathsf{Set}^{\mathsf{Cxt}}$. Explicitly, an object P of the category $\mathsf{Set}^{\mathsf{Cxt}}$ is a Cxt-indexed family of sets: for any context Γ, $P\ \Gamma$ is a set[1]. A morphism f between P and Q in $\mathsf{Set}^{\mathsf{Cxt}}$ is a natural transformation, i.e. a Cxt-indexed family of functions: for any context Γ, f is a function between $P\ \Gamma$ and $Q\ \Gamma$. We typically omit the index Γ, and simply write $f : P\ \Gamma \to Q\ \Gamma$. The set of morphisms between P and Q is denoted $P \dot{\to} Q$.

The category $\mathsf{Set}^{\mathsf{Cxt}}$ is *monoidal biclosed*, with unit, tensor and internal homs given by[2]:

$$\widehat{\mathsf{I}}\ \Gamma = (\Gamma = \langle\rangle) \qquad (P \mathbin{\widehat{\otimes}} Q)\ \Gamma = \{\Gamma_0, \Gamma_1 : \mathsf{Cxt}\} \times \{\Gamma = \Gamma_0, \Gamma_1\} \times P\ \Gamma_0 \times Q\ \Gamma_1$$
$$(P \mathbin{\widehat{\diagup}} Q)\ \Gamma = \{\Delta : \mathsf{Cxt}\} \to Q\ \Delta \to P\ (\Gamma, \Delta) \qquad (Q \mathbin{\widehat{\diagdown}} P)\ \Gamma = \{\Delta : \mathsf{Cxt}\} \to Q\ \Delta \to P\ (\Delta, \Gamma)$$

The tensor and internal homs in $\mathsf{Set}^{\mathsf{Cxt}}$ form two adjunctions: $-\mathbin{\widehat{\otimes}} Q \dashv Q \mathbin{\widehat{\diagdown}} -$ and $P \mathbin{\widehat{\otimes}} - \dashv - \mathbin{\widehat{\diagup}} P$.

The monoidal biclosed category $\mathsf{Set}^{\mathsf{Cxt}}$ is not completely suitable for the construction of an algorithm satisfying the NbE specification (more on why later). In analogy with the case of intuitionistic propositional logic with falsity and disjunction [1], we introduce a monad T on $\mathsf{Set}^{\mathsf{Cxt}}$. For each presheaf P and context Γ, the elements of the set $T\ P\ \Gamma$ are inductively generated by the following constructors:

$$\frac{P\ \Gamma}{T\ P\ \Gamma}\ \eta \qquad \frac{\Gamma \Downarrow A \otimes B \quad T\ P\ (\Delta_0, A, B, \Delta_1)}{T\ P\ (\Delta_0, \Gamma, \Delta_1)}\ E^T_{\otimes}$$

[1] We think of Cxt as a discrete category, which is why there is no mention of P's action of morphisms.

[2] We use Agda notation for the dependent sum and dependent product operations: $(x : A) \times B\ x$ and $(x : A) \to B\ x$ stand for $\sum_{x \in A} B\ x$ and $\prod_{x \in A} B\ x$ respectively. Curly brackets indicate implicit arguments.

Interpretation of Syntax. Each formula A is interpreted as a presheaf $[\![A]\!]$:

$$[\![p]\!] = - \Uparrow p \qquad [\![B \mathbin{/} A]\!] = [\![B]\!] \mathbin{\widehat{/}} [\![A]\!] \qquad [\![A \otimes B]\!] = T\ ([\![A]\!] \mathbin{\widehat{\otimes}} [\![B]\!]) \qquad [\![A \mathbin{\backslash} B]\!] = [\![A]\!] \mathbin{\widehat{\backslash}} [\![B]\!]$$

Notice the application of the monad T in the interpretation of $\otimes$. The interpretation of formulae extends to contexts via the monoidal structure of $\mathsf{Set}^{\mathsf{Cxt}}$: $[\![\langle\rangle]\!] = \widehat{\mathsf{I}}$ and $[\![A, \Gamma]\!] = [\![A]\!] \mathbin{\widehat{\otimes}} [\![\Gamma]\!]$. Each derivation $t : \Gamma \vdash A$ in $\mathbf{L}$ is interpreted as a natural transformation $[\![t]\!] : [\![\Gamma]\!] \mathrel{\dot{\to}} [\![A]\!]$[3]:

$$\begin{array}{lcllcl}
[\![\mathsf{ax}]\!]\ x & = & x & [\![I_\otimes\ t\ u]\!]\ (\gamma, \delta) & = & \eta\ ([\![t]\!]\ \gamma, [\![u]\!]\ \delta) \\
[\![E_{/}\ t\ u]\!]\ (\gamma, \delta) & = & [\![t]\!]\ \gamma\ ([\![u]\!]\ \delta) & [\![I_{/}\ t]\!]\ \gamma & = & \lambda x.\ [\![t]\!]\ (\gamma, x) \\
[\![E_{\backslash}\ t\ u]\!]\ (\gamma, \delta) & = & [\![u]\!]\ \delta\ ([\![t]\!]\ \gamma) & [\![I_{\backslash}\ t]\!]\ \gamma & = & \lambda x.\ [\![t]\!]\ (x, \gamma) \\
[\![E_\otimes\ t\ u]\!]\ (\delta_0, \gamma, \delta_1) & = & \multicolumn{4}{l}{\mathsf{run}\ (T\ [\![u]\!]\ (\mathsf{rmst}\ (\mathsf{lmst}\ (\delta_0, [\![t]\!]\ \gamma), \delta_1)))}
\end{array} \tag{1}$$

The map $\mathsf{run}_A : T\ [\![A]\!] \mathrel{\dot{\to}} [\![A]\!]$ in (1) is a constructible natural transformation for "running" the monad T on interpreted formulae, while rmst and lmst are the left and right strengths of T wrt. the monoidal structure $(\widehat{\mathsf{I}}, \widehat{\otimes})$.

The Normalization Function. The last phase of the NbE procedure is the extraction of normal forms from elements of the Kripke model. Concretely, this correponds to the construction of a *reification* function $\downarrow_A$: $[\![A]\!] \mathrel{\dot{\to}} - \Uparrow A$. In order to deal with the cases of the mixed-variance connectives $/$ and $\backslash$, it is necessary to simultaneously define reification together with a *reflection* procedure $\uparrow_A$: $- \Downarrow A \mathrel{\dot{\to}} [\![A]\!]$, embedding neutrals in the presheaf model. This is the crucial point were the interpretation of the tensor product $[\![A \otimes B]\!] = T\ ([\![A]\!] \mathbin{\widehat{\otimes}} [\![B]\!])$ works, while a naïve interpretation $[\![A \otimes B]\!] = [\![A]\!] \mathbin{\widehat{\otimes}} [\![B]\!]$ without the application of the monad T would fail. With the latter interpretation, $\uparrow_{A \otimes B}\ t$ would be required to have type $([\![A]\!] \mathbin{\widehat{\otimes}} [\![B]\!])\ \Gamma$, which in turn will force us to split the context $\Gamma = \Gamma_0, \Gamma_1$ and provide elements of type $[\![A]\!]\ \Gamma_0$ and $[\![B]\!]\ \Gamma_1$. But this split is generally impossible to achieve, e.g. the neutral t could be a variable of the form $\mathsf{ax} : A \otimes B \Downarrow A \otimes B$. This problematic splitting is avoided through the application of the constructor $E^T_\otimes$. This seems to be a general pattern for all positive connectives, e.g. consider the case of falsity and disjunction in intuitionistic propositional logic [1, 4].

$$\begin{array}{lcllcl}
\downarrow_p\ t & = & t & \uparrow_p t & = & \mathsf{sw}_{\Downarrow} t \\
\downarrow_{B/A}\ t & = & I_{/}\ (\downarrow_B\ (t\ (\uparrow_A\ \mathsf{ax}))) & \uparrow_{B/A} t & = & \lambda x.\ \uparrow_B\ (E_{/}\ t\ (\downarrow_A\ x)) \\
\downarrow_{A\backslash B}\ t & = & I_{\backslash}\ (\downarrow_B\ (t\ (\uparrow_A\ \mathsf{ax}))) & \uparrow_{A\backslash B} t & = & \lambda x.\ \uparrow_B\ (E_{\backslash}\ (\downarrow_A\ x)\ t) \\
\downarrow_{A\otimes B}\ t & = & \mathsf{run}^{\Uparrow}\ (T\ (\lambda(x, y).\ I_\otimes\ (\downarrow_A\ x)\ (\downarrow_B\ y))\ t) & \uparrow_{A\otimes B} t & = & E^T_\otimes\ t\ (\eta\ (\uparrow_A \mathsf{ax}, \uparrow_B \mathsf{ax}))
\end{array}$$

The reflection function $\uparrow_A$ can also be used for the definition of an element $\mathsf{fresh}_\Gamma : [\![\Gamma]\!]\ \Gamma$, for each context Γ.

The normalization function $\mathsf{nbe} : \Gamma \vdash A \to \Gamma \Uparrow A$ is then definable as the reification of the interpretation of a derivation $t : \Gamma \vdash A$ in the Kripke model: $\mathsf{nbe}\ t = \downarrow_A\ ([\![t]\!]\ \mathsf{fresh}_\Gamma)$. Here we consider the interpretation $[\![t]\!] : [\![\Gamma]\!]\ \Gamma \to [\![A]\!]\ \Gamma$, which we can apply to $\mathsf{fresh}_\Gamma : [\![\Gamma]\!]\ \Gamma$. Intuitively, the element fresh_Γ gives an intepretation to all the free variables in t. Since $[\![t]\!] = [\![u]\!]$ for all $t \sim u$, then the function nbe sends $\sim$-related derivations in $\mathbf{L}$ to equal derivations, i.e. $\mathsf{nbe}\ t = \mathsf{nbe}\ u$ whenever $t \sim u$.

A normalization function internal to $\mathbf{L}$ can be defined by postcomposing nbe with $\mathsf{emb}_{\Uparrow}$. The resulting procedure is correct whenever each derivation t is $\sim$-equivelent to its normal form $\mathsf{emb}_{\Uparrow}\ (\mathsf{nbe}\ t)$. Following the standard NbE strategy [3, 1], correctness can be proved using logical relations.

[3] In the $E_\otimes$ case, applications of the natural isomorphisms of unitality and associativity of $(\widehat{\mathsf{I}}, \widehat{\otimes})$ are omitted.

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