On the axiomatizability of priority III: The return of sequential composition*

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Abstract. Aceto et al., proved that, over the process algebra BCCSP with the *priority operator* of Baeten, Bergstra and Klop, the equational theory of *order-insensitive bisimilarity* is not finitely based. However, it has been noticed that by substituting the action prefixing operator of BCCSP with BPA's *sequential composition*, the infinite family of equations used to show that non-finite axiomatizability result could be proved by a finite collection of sound equations. That observation left as an *open question* the existence of a finite axiomatization for order-insensitive bisimilarity over BPA with the priority operator. In this paper we provide a negative answer to this question. We prove that, in the presence of at least two actions, order-insensitive bisimilarity *is not finitely based* over BPA with priority.

1 Introduction

Process algebras are a classic tool for reasoning about the behaviour of concurrent and distributed systems. One important aspect of systems is that of a priority between actions. For example, an interrupt or shutdown action may be needed when a system deadlocks or starts exhibiting erroneous behaviour, and likewise a scheduler needs to assign priority to actions based on its scheduling policy. There have been various attempts in the literature on process algebras at taking priority into consideration, see e.g., [10] for an overview. Here we consider the approach taken in [5], where a priority operator is introduced. This operator is based on an order between the actions that are available to a system, and only allows an action to be performed if no other action with a higher priority is possible at the given moment. For example, this allows interrupts to be given priority over all other actions.

It was shown in [5, 7] that the priority operator admits a finite, ground-complete equational axiomatization modulo bisimilarity, where ground-complete means that the given set of axioms can prove all sound equations where the process terms do not have variables. This result holds when the set of possible actions is finite. For an infinite set of actions, it was proved in [2] that the priority operator admits no finite equational axiomatization in the setting of the process algebra BCCSP, which consists of basic

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operators from CCS [15, 16] and CSP [12, 13]. Furthermore, a specific priority order was exhibited for which no finite equational ground-complete axiomatization exists.

The results mentioned so far consider only a single, given priority order. One may expect that if we consider order-insensitive bisimilarity, i.e. processes are bisimilar if they are bisimilar for every priority order, then there are no sound equations of interest that involve the priority operator. However, as shown in [3], this is not the case, and there is no finite, ground-complete equational axiomation modulo order-insensitive bisimilarity. In contrast to the previous result for fixed priority orders, this result holds as long as there are at least two actions. We note that if this assumption is not satisfied, then the priority operator becomes redundant. However, it was remarked in [3] that the infinite family of equations that was used to show the above mentioned negative result could be replaced by a single equation if one allows general sequential composition rather than just action prefixing, thus invalidating the argument in this extended setting.

In this paper, we therefore consider the process algebra BPA [8], which is essentially an extension of BCCSP with general sequential composition. We show that, also in this setting, the priority order admits no finite, ground-complete equational axiomatization modulo order-insensitive bisimilarity whenever there are at least two possible actions. In order show this result, we make use of the notion of a configuration from [4], which allows us to reason about the behaviour of an instantiation of a variable along its execution. The key part of the argument is to consider an infinite family of sound equations relating variable-free terms where, at each step, the process has the possibility of terminating, thus ensuring that the equations cannot be written as a sequential composition, and then to show that this specific family of equations cannot be proved from a finite number of axioms.

Outline of the paper: We start by reviewing background notions in Section 2. Section 3 comes with technical results necessary to reason on the semantics of open process terms. In Section 4 we provide the properties necessary to ensure that order-insensitive bisimilarity behaves coinductively. Our main result is in Section 5 where we prove that the order-insensitive bisimilarity is not finitely based over BPA with the priority operator. Finally, we draw some conclusions and discuss future work in Section 6.

Due to the space limitations, the full proofs of our technical results can be found in the Appendix.

2 Background

In this section we review some preliminary notions on operational semantics and equational logic. Since our work naturally builds on [3, 4] we will use the notation from those papers as much as possible.

BPA$_\Theta$: syntax and semantics The syntax of *process terms* in BPA$_\Theta$, namely BPA [8] enriched with the priority operator [5], is generated by the following grammar

$$t ::= a \mid x \mid t \cdot t \mid t + t \mid \Theta(t),$$

with a ranging over a set of actions $\mathcal{A}$, x ranging over a countably infinite set of variables $\mathcal{V}$ and t ranging over process terms. A process term is *closed* if no variable occurs

$$(r_1)\ \frac{}{a \xrightarrow{a}_{>} \surd} \qquad (r_2)\ \frac{p \xrightarrow{a}_{>} \surd}{p \cdot q \xrightarrow{a}_{>} q} \qquad (r_3)\ \frac{p \xrightarrow{a}_{>} p'}{p \cdot q \xrightarrow{a}_{>} p' \cdot q}$$

$$(r_4)\ \frac{p \xrightarrow{a}_{>} \surd}{p + q \xrightarrow{a}_{>} \surd} \qquad (r_5)\ \frac{q \xrightarrow{a}_{>} \surd}{p + q \xrightarrow{a}_{>} \surd} \qquad (r_6)\ \frac{p \xrightarrow{a}_{>} p'}{p + q \xrightarrow{a}_{>} p'} \qquad (r_7)\ \frac{q \xrightarrow{a}_{>} q'}{p + q \xrightarrow{a}_{>} q'}$$

$$(r_8)\ \frac{p \xrightarrow{a}_{>} \surd \quad \forall\, b > a\,.\, p \not\xrightarrow{b}_{>}}{\Theta(p) \xrightarrow{a}_{>} \surd} \qquad (r_9)\ \frac{p \xrightarrow{a}_{>} p' \quad \forall\, b > a\,.\, p \not\xrightarrow{b}_{>}}{\Theta(p) \xrightarrow{a}_{>} \Theta(p')}$$

Table 1: Operational semantics of processes in BPA$_{\Theta}$.

in it. We shall refer to closed process terms simply as *processes*. We let $\mathbf{P}$ denote the set of BPA$_{\Theta}$ processes and let $p, q, \dots$ range over it.

We use the *structural operational semantics* framework [18] to equip processes with a semantics. A *literal*, or *open transition*, is an expression of the form $t \xrightarrow{a} t'$ for some process terms t, t' and action $a \in \mathcal{A}$. It is *closed* if both t, t' are processes. The inference rules for *sequential composition* $\cdot$, alternative *nondeterministic choice* $+$ and *priority* Θ are reported in Table 1. We remark that the semantics of Θ is based on a strict partial order $>$ on $\mathcal{A}$, called the *priority order*, which justifies the parametrization of the derived transition relation with respect to $>$. For simplicity, given $a, b \in \mathcal{A}$, we write $a > b$ for $(a, b) \in >$. To deal with sequential composition in the absence of deadlock and empty process (see, e.g., [8, 19]), we introduce the *termination predicate* $\rightarrow_{>} \surd \subseteq \mathbf{P} \times \mathcal{A}$. Intuitively, $t \xrightarrow{a}_{>} \surd$ means that t can terminate successfully in one step by performing action a. A *substitution* σ is a mapping from variables to process terms. It extends to process terms, literals and rules in the usual way and it is *closed* if it maps every variable to a process. We denote by $\sigma[x \mapsto u]$ the substitution that maps each occurrence of the variable x into the process term u and behaves like σ over all other variables. The inference rules in Table 1 induce a unique supported model [11] corresponding to the $\mathcal{A}$-labelled transition system $(\mathbf{P}, \mathcal{A}, \rightarrow_{>}, \rightarrow_{>} \surd)$ whose transition relation $\rightarrow_{>}$ (respectively, predicate $\rightarrow_{>} \surd$) contains exactly the closed literals (respectively, predicates) that can be derived by structural induction over processes using the rules in Table 1.

As usual, we write $p \xrightarrow{a}_{>} p'$ for $(p, a, p') \in \rightarrow_{>}$, $p \rightarrow_{>} p'$ if $p \xrightarrow{a}_{>} p'$ for some $a \in \mathcal{A}$, and $p \not\xrightarrow{a}_{>}$ if there is no p' s.t. $p \xrightarrow{a}_{>} p'$. For $k \in \mathbb{N}$, we write $p \rightarrow_{>}^{k} p'$ if there are $p_0, \dots, p_k$ s.t. $p = p_0 \rightarrow_{>} \cdots \rightarrow_{>} p_k = p'$. Furthermore, for a sequence of actions $s = a_1 \dots a_n$, we write $p \xrightarrow{s}_{>} p'$ to mean that $p \xrightarrow{a_1}_{>} p_1 \xrightarrow{a_2}_{>} \cdots p_{n-1} \xrightarrow{a_n}_{>} p'$ for some processes $p_1, \dots, p_{n-1}$.

We associate two classic measures with each process: its *depth* and its *norm*. As usual, they express, respectively, the length of a *longest* and a *shortest* sequence of transitions that are enabled for the process (we refer the interested reader to Appendix A for the formal definitions and their relations with the priority order).

For $p \in \mathbf{P}$, the set of *initial actions* of p with respect to $>$ is defined as

$$\mathcal{A}_{>}(p) = \{a \mid p \xrightarrow{a}_{>} p', p' \in \mathbf{P}\} \cup \{a \mid p \xrightarrow{a}_{>} \surd\}.$$

We extend this notion to sequences of transitions by letting $\mathcal{A}^k_>(p) = \bigcup_{p \longrightarrow^k_> p'} \mathcal{A}_>(p')$ and $\mathcal{A}^*_>(p) = \bigcup_{k \in \mathbb{N}} \mathcal{A}^k_>(p)$ be, respectively, the set of actions that are enabled with respect to $>$ at depth k and at some depth. We say that action a is *maximal* with respect to $>$ if there is no $b \in \mathcal{A}$ s.t. $b > a$. We can restrict this notion to the set of actions that are enabled for a process. Given a process p, we say that an action $a \in \mathcal{A}^*_>(p)$ is *maximal in* p, or *locally maximal*, with respect to $>$ if there is no $b \in \mathcal{A}^*_>(p)$ s.t. $b > a$. If $\mathcal{A}^*_>(p) = \{a\}$ then a is locally maximal with respect to $>$.

Order-insensitive bisimulation With the priority operator, the set of transitions that are enabled for each process depends on the considered priority order on $\mathcal{A}$. Therefore, any bisimulation relation over BPA$_\Theta$ processes will also depend on the priority order. In [3], along all such bisimulations, the authors introduced the notion of *order-insensitive bisimilarity*, $\underline{\leftrightarrow}_*$, formally defined as the intersection over all priority orders of the related bisimulation relations. Since $\underline{\leftrightarrow}_*$ disregards the particular order that is considered, it can be used to study general properties of processes and thus developing a general equational theory for BPA$_\Theta$.

Definition 1 (Order-insensitive bisimulation, [3]). *Let $>$ be any priority order. A binary symmetric relation $\mathcal{R} \subseteq \mathbf{P} \times \mathbf{P}$ is a* bisimulation with respect to $>$ *if whenever $p\mathcal{R}q$ then (i) $\forall\, p \xrightarrow{a}_> p'$ there is $q \xrightarrow{a}_> q'$ s.t. $p'\mathcal{R}q'$, and (ii) $\forall\, p \xrightarrow{a}_> \surd$ also $q \xrightarrow{a}_> \surd$ holds. We say that p, q are* bisimilar with respect to $>$, *denoted by $p \,\underline{\leftrightarrow}_>\, q$, if $p\mathcal{R}q$ holds for some bisimulation $\mathcal{R}$ with respect to $>$. We say that p, q are* order-insensitive bisimilar*, denoted by $p \,\underline{\leftrightarrow}_*\, q$, if $p \,\underline{\leftrightarrow}_>\, q$ holds for all priority orders.*

It is not hard to prove that, since the inference rules in Table 1 respect the GSOS format [9], $\underline{\leftrightarrow}_>$ and $\underline{\leftrightarrow}_*$ are congruences over BPA$_\Theta$ processes. However, as discussed in [3], $\underline{\leftrightarrow}_*$ does not inherit the coinductive nature of bisimilarity. Consider, for instance, the processes $p = a \cdot b + a \cdot c + a \cdot (b + c)$ and $q = p + a \cdot \Theta(b + c)$. Notice that if $b > c$ then $a \cdot \Theta(b + c) \,\underline{\leftrightarrow}_>\, a \cdot b$, if $c > b$ then $a \cdot \Theta(b + c) \,\underline{\leftrightarrow}_>\, a \cdot c$, and if b, c are incomparable with respect to $>$ then $a \cdot \Theta(b + c) \,\underline{\leftrightarrow}_>\, a \cdot (b + c)$. Therefore, we have that $p \,\underline{\leftrightarrow}_*\, q$. However, $q \xrightarrow{a}_> \Theta(b+c)$ for each order $>$, but there is no p' s.t. $p \xrightarrow{a}_> p'$ and $p' \,\underline{\leftrightarrow}_*\, \Theta(b + c)$.

Henceforth, whenever $>$ is the empty order, we simply omit the subscript.

Equational logic An *axiom system* E is a collection of *process equations* $t \approx u$ over the language BPA$_\Theta$, such as those presented in Table 2. An equation $t \approx u$ is *derivable* from an axiom system E, notation $E \vdash t \approx u$, if there is an *equational proof* for it from E, namely if it can be inferred from the axioms in E using the *rules* of *equational logic*, which are reflexivity, symmetry, transitivity, substitution and closure under BPA$_\Theta$ contexts. We refer the interested reader to Table 4 in Appendix D.4 for a complete presentation of such rules.

The process equation $t \approx u$ is said to be *sound* with respect to $\underline{\leftrightarrow}_*$ if $\sigma(t) \,\underline{\leftrightarrow}_*\, \sigma(u)$ for all closed substitutions σ. For simplicity, if $t \approx u$ is sound, then we write $t \,\underline{\leftrightarrow}_*\, u$. An axiom system is *sound* modulo $\underline{\leftrightarrow}_*$ if and only if all of its equations are sound modulo $\underline{\leftrightarrow}_*$. Conversely, we say that E is *ground-complete* modulo $\underline{\leftrightarrow}_*$ if $p \underline{\leftrightarrow}_* q$ implies $E \vdash p \approx q$ for all processes p, q. We say that $\underline{\leftrightarrow}_*$ is *finitely based*, if there is a *finite*

C1	$x + y \approx y + x$	S1	$(x \cdot y) \cdot z \approx x \cdot (y \cdot z)$
C2	$(x + y) + z \approx x + (y + z)$	S2	$(x + y) \cdot z \approx (x \cdot z) + (y \cdot z)$
C3	$x + x \approx x$		
P1	$\Theta(\Theta(x) + y) \approx \Theta(x + y)$		
P2	$\Theta(x) + \Theta(y) \approx \Theta(x) + \Theta(y) + \Theta(x + y)$		
P3	$\Theta(x \cdot y) \approx \Theta(x) \cdot \Theta(y)$		
P4	$\Theta(x \cdot y + x \cdot z + w) \approx \Theta(x \cdot y + w) + \Theta(x \cdot z + w)$		

Table 2: Some axioms of BPA$_\Theta$.

axiom system E s.t. $E \vdash t \approx u$ iff $t \underline{\leftrightarrow}_* u$. Finally, notice that the notion of depth can be extended to equations by letting $\mathrm{depth}\,(t \approx u) = \max\{\mathrm{depth}\,(t)\,, \mathrm{depth}\,(u)\}$.

3 Relation between open and closed operational behaviour

Our purpose in the remainder of this paper is to verify whether the axiomatization for order-insensitive bisimilarity is finitely based over BPA$_\Theta$. To address this question it is fundamental to establish a correspondence between the behavior of open terms and the semantics of their closed instances, with a special focus on the role of variables. In fact, the equational theory is defined over process terms, whereas the semantic properties can be verified only on their closed instances. In this section, we provide the notions and theoretical results necessary to establish the desired behavioral correspondence.

3.1 From open to closed transitions…

Assume a term t, a closed substitution σ, a process p, an action a and a priority order $>$. We aim at investigating how to derive a transition of the form $\sigma(t) \xrightarrow{a}_> p$, as well as a predicates $\sigma(t) \xrightarrow{a}_> \surd\!\!\!\surd$, from the behavior of t and of $\sigma(x)$ for each variable x occurring in t. In particular we are interested in relating the *initial* behavior of $\sigma(t)$ with the behavior of closed instances of variables occurring in it.

The simplest case is a direct application of the operational semantics in Table 1: if action a is maximal with respect to $>$, then $\sigma(t) \xrightarrow{a}_> p$ can be inferred directly by $t \xrightarrow{a}_> t'$, for some term t' with $\sigma(t') = p$. Similarly, for transition predicates.

Lemma 1. *Let t, t' be process terms, let a be an action with maximal priority with respect to $>$. Then for all substitutions σ it holds that:*

1. *If $t \xrightarrow{a}_> \surd\!\!\!\surd$ then $\sigma(t) \xrightarrow{a}_> \surd\!\!\!\surd$.*
2. *If $t \xrightarrow{a}_> t'$ then $\sigma(t) \xrightarrow{a}_> \sigma(t')$.*

Next we deal with variables. It may be the case, for instance, that the term t is of the form $t = x \cdot u$ for some term u. Clearly, the behavior of $\sigma(t)$, and thus the derivation of $\sigma(t) \xrightarrow{a}_> p$, will depend on the behavior of $\sigma(x)$. However, we remark that there is not a unique derivation method. The set of initial actions of $\sigma(t)$ does not depend, in general, solely on those of $\sigma(x)$, but also on the structure of the process into which x is mapped, and by the occurrence of x in t. For instance, for $t = x \cdot u$ we can distinguish two main situations:

(I) Suppose $\sigma(x) = a$, so that $\sigma(x) \xrightarrow{a}_{>} \surd$. This would give $\sigma(t) \xrightarrow{a}_{>} p$ for $p = \sigma(u)$, namely p is a closed instance of a subterm of t. Therefore, the transition for $\sigma(t)$ could be expressed in term of a closed instance of an open transition for t, as $t \rightarrow_{>} t'$. However, notice that the action that is performed cannot be obtained from the term t as it depends solely on the substitution applied to x. Hence, we will need a formal way to express that the label of the transition depends on x.

(II) Suppose $\sigma(x) = a \cdot b$, so that $\sigma(x) \xrightarrow{a}_{>} b$. Clearly, $\sigma(t)$ will have to mimic such behavior, and thus $\sigma(t) \xrightarrow{a}_{>} p$ with $p = b \cdot \sigma(u)$. Notice that process p subsumes *what's left* of the behavior of $\sigma(x)$. Then the transition for $\sigma(t)$ cannot be inferred by a closed substitution instance of an open transition of the form $t \xrightarrow{a}_{>} t'$, since the structure of t' cannot be known until the substitution $\sigma(x)$ has occurred. Hence, we will need a formal way to express that to reach a subterm of t we need to follow a sequence of transitions performed by x.

For a formal development of the analysis in the above-mentioned cases, we exploit the method proposed in [4] and provide an auxiliary operational semantics tailored for expressing the behavior of process terms resulting from that of closed substitution instances for their variables.

Firstly we introduce the notion of *configuration* over BPA$_{\Theta}$ terms, which stems from [4]. Configurations are syntactic terms defined over a set of variables $\mathcal{V}_{\mathrm{d}} = \{x_d \mid x \in \mathcal{V}\}$ disjoint from $\mathcal{V}$ and BPA$_{\Theta}$ terms. Briefly, we use the variable x_d to express that the closed instance of x has started its execution, but has not terminated yet.

Definition 2 (BPA$_{\Theta}$ configuration). *The collection of BPA$_{\Theta}$ configurations is given by the grammar:*

$$c ::= t \mid x_d \mid c \cdot t \mid \Theta(c),$$

where t is a BPA$_{\Theta}$ term and $x_d \in \mathcal{V}_{\mathrm{d}}$.

Notice that the grammar above guarantees that each configuration contains at most one occurrence of a variable in $\mathcal{V}_{\mathrm{d}}$, say x_d, and if such occurrence is in the scope of sequential composition, then x_d must occur as the first symbol in the composition.

Define the set of variable labels $\mathcal{V}_{\mathrm{s}} = \{x_s \mid x \in \mathcal{V}\}$, disjoint from $\mathcal{V}$ and assume any priority order $>$. We then introduce two auxiliary relations $\xrightarrow{x_s}_{>}$, $\xrightarrow{x}_{>}$, and the auxiliary predicate $\xrightarrow{x}_{>} \surd$, whose operational semantics is given in Table 3, and that allow us to express how the initial behavior of a term can be derived from that of the variables occurring in it. Informally, the labels allow us to identify the variable that is inducing that particular transition. Moreover, $t \xrightarrow{x}_{>} t'$ (resp. $t \xrightarrow{x}_{>} \surd$) is used to describe the derivation of a transition $\sigma(t) \xrightarrow{a}_{>} \sigma(t')$ (resp. of the validity of predicate $\sigma(t) \xrightarrow{a}_{>} \surd$) in the case described in item (I) above: $\sigma(x)$ performs action a and terminates, and in doing so it enables the execution of the subprocess $\sigma(t')$, besides triggering the a-transition. Conversely, the auxiliary transition $t \xrightarrow{x_s}_{>} c$ is used to deal with the case described in item (II) above: $\sigma(x)$ started its execution, but since it has

$$
(a_1)\ \frac{}{x \xrightarrow{x_s}_{>} x_d} \qquad (a_2)\ \frac{}{x \xrightarrow{x}_{>} \surd}
$$

$$
(a_3)\ \frac{t \xrightarrow{x_s}_{>} c}{t \cdot u \xrightarrow{x_s}_{>} c \cdot u} \qquad (a_4)\ \frac{t \xrightarrow{x}_{>} t'}{t \cdot u \xrightarrow{x}_{>} t' \cdot u} \qquad (a_5)\ \frac{t \xrightarrow{x}_{>} \surd}{t \cdot u \xrightarrow{x}_{>} u}
$$

$$
(a_6)\ \frac{t \xrightarrow{x_s}_{>} c}{t + u \xrightarrow{x_s}_{>} c} \qquad (a_7)\ \frac{t \xrightarrow{x}_{>} t'}{t + u \xrightarrow{x}_{>} t'} \qquad (a_8)\ \frac{t \xrightarrow{x}_{>} \surd}{t + u \xrightarrow{x}_{>} \surd}
$$

$$
(a_9)\ \frac{t \xrightarrow{x_s}_{>} c}{\Theta(t) \xrightarrow{x_s}_{>} \Theta(c)} \qquad (a_{10})\ \frac{t \xrightarrow{x}_{>} t'}{\Theta(t) \xrightarrow{x}_{>} \Theta(t')} \qquad (a_{11})\ \frac{t \xrightarrow{x}_{>} \surd}{\Theta(t) \xrightarrow{x}_{>} \surd}
$$

Table 3: Inference rules for the auxiliary transition relations. The symmetric versions of rules a_6–a_8 have been omitted.

not terminated yet, there is no subterm of t that can be used as target of the open transition. Thus, we use the configuration c to store the *yet-to-terminate* behavior of $\sigma(x)$. In the case of item (II), we would have $c = x_d \cdot u$, and since $\sigma(x) \xrightarrow{a}_{>} b$, we would let $\sigma[x_d \mapsto b](c) = b \cdot \sigma(u)$.

The following lemma formalizes the intuitions above. To avoid conflicts with any possible occurrence of the priority operator, we focus only on transitions labeled with actions that are (locally) maximal with respect to the chosen priority operator $>$. This type of transition will be sufficient for our purposes in the rest of the paper.

Lemma 2. *Let t be a process term, x a variable, σ a substitution and $a \in \mathcal{A}$ be maximal with respect to $>$. Then:*

1. *If $t \xrightarrow{x}_{>} \surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd$, then $\sigma(t) \xrightarrow{a}_{>} \surd$.*
2. *If $t \xrightarrow{x}_{>} t'$ and $\sigma(x) \xrightarrow{a}_{>} \surd$, then $\sigma(t) \xrightarrow{a}_{>} \sigma(t')$.*
3. *If $t \xrightarrow{x_s}_{>} c$ and $\sigma(x) \xrightarrow{a}_{>} p$ for some process p, then $\sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$.*

We will sometimes need to extend the third case of Lemma 2 to sequences of transitions. The following lemma allows us to do so by proceeding inductively.

Lemma 3. *Let σ be a closed substitution. If $t \xrightarrow{x_s}_{>} c$ and $\sigma(x) \to^n_{>} p$ is such that all actions taken along the transitions from $\sigma(x)$ to p are maximal with respect to $>$, then $\sigma(t) \to^n_{>} \sigma[x_d \mapsto p](c)$.*

3.2 . . . and back again

So far we have provided a way to derive the initial behavior of a term from the open transitions available for it, in particular when determined by variables. Our aim is now to obtain a converse result: knowing that $\sigma(t) \xrightarrow{a}_{>} p$, we want to derive its possible sources in the behavior of t and of the closed instances of the variables occurring in t.

Firstly, we remark that, since before we were working with process terms, no occurrence of a priority operator due to substitutions could have been foreseen. Therefore, to avoid conflicts, we have limited our attention to actions that were (locally) maximal with respect to the considered priority order. However, we now start from $\sigma(t)$ and therefore we can properly relate the behavior of variables to their potential occurrence in the scope of a priority operator. To this end, we introduce an extended version of the relation $\triangleleft_l$ from [3], relating a variable x and a term t with respect to the natural number $l \in \mathbb{N}$, notation $x \triangleleft_l t$, if x occurs *unguarded* in t in the scope of l-nested applications of the priority operator.

Definition 3 (Relation $\triangleleft_l$). *The relations $\triangleleft_l$, for $l \in \mathbb{N}$, between variables and terms are defined as the least relations satisfying the following constraints:*

1. *$x \triangleleft_0 x$;*
2. *if $x \triangleleft_l t$ then $x \triangleleft_l t + u$ and $x \triangleleft_l u + t$;*
3. *if $x \triangleleft_l t$ then $x \triangleleft_l t \cdot t'$;*
4. *if $x \triangleleft_l t$ then $x \triangleleft_{l+1} \Theta(t)$.*

If $x \triangleleft_l t$, for some $l \in \mathbb{N}$, we shall say that x is enabled *in t.*

As stated by the following lemma, there is a close relation between x being enabled in t and the auxiliary transition $t \xrightarrow{x_s}_{>} c$. We write $t = t_1 \odot t_2$ to mean that either $t = t_1$ or $t = t_1 \cdot t_2$, i.e., t_1 may possibly be sequentially followed by t_2.

Lemma 4. *Assume a variable x, a term t and a natural number $l \in \mathbb{N}$. Then, $x \triangleleft_l t$ if and only if $t \xrightarrow{x_s}_{>} \odot^l(x_d)$ where $\odot^l(x_d)$ is a configuration of the form*

$$\odot^l(x_d) = \underbrace{\Theta(\cdots\Theta}_{l\ times}(x_d \odot t_{l+1}) \odot t_l) \dots) \odot t_1.$$

The notation $\odot^l(x_d)$ abstracts away from the trailing $t_{l+1}, \dots, t_1$. This choice is due to mere simplification purposes and does not impact the technical development of our results. In fact, the beaviour of the terms $t_{l+1}, \dots, t_1$ and their closed instances will never play a role in such results, as only the behavioural properties of closed instances of x_d will be of interest. We remark also that $\odot^0(x_d)$ denotes a configuration containing an occurrence of x_d which does not occur in the scope of a priority operator.

We are now ready to derive the behavior of the term t and that of the closed instances of the variables occurring in t, from the transitions enabled for $\sigma(t)$.

Proposition 1. *Let t be a process term, σ a closed substitution, a an action and p a process. Then:*

1. *If $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\surd$ then*
   (a) *either $t \xrightarrow{a}_{>} \surd\!\!\surd$;*
   (b) *or there is a variable x such that $t \xrightarrow{x}_{>} \surd\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$.*
2. *If $\sigma(t) \xrightarrow{a}_{>} p$ then one of the following applies:*
   (a) *there is a process term t' such that $t \xrightarrow{a}_{>} t'$ and $\sigma(t') = p$;*
   (b) *there is a process term t' and a variable x such that $t \xrightarrow{x}_{>} t'$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ and $\sigma(t') = p$;*

(c) *there is a variable x, a natural number $l \in \mathbb{N}$, and a process q such that $t \xrightarrow{x_s}_{>} \odot^l(x_d)$, $\sigma(x) \xrightarrow{a}_{>} q$ and $\odot^l(q) = p$.*

Assume a process term t and suppose that $\mathrm{depth}\,(t) = k$ for some $k \in \mathbb{N}$. Clearly, given any closed substitution σ we will have that $\mathrm{depth}\,(\sigma(t)) = n$ for some $n \geq k$. In particular, whenever n is *strictly* greater than k we can infer that at least one variable occurring in t has been mapped into a process defined via the sequential composition operator. To conclude this section, we extend Proposition 1 to sequences of transitions of an arbitrary length.

To this end, we introduce the following notation: let $w \in (\mathcal{A} \cup \mathcal{V})^*$ be a string $w = \alpha_1 \ldots \alpha_h$ in which each α_i can be either an action or a variable. Then, given a substitution σ, we write $t \xrightarrow{s_1 \ldots s_h}_{>,w} t'$ if there are process terms $t_0, \ldots, t_h$ such that $t = t_0$, $t' = t_h$, and, for all $i \in \{1, \ldots, h\}$,

- $s_i \in \mathcal{A}^*$;
- if $\alpha_i \in \mathcal{V}$, then $t_{i-1} \xrightarrow{s_i}_{>} t_i$ and $\sigma(\alpha_i) \xrightarrow{s_i}_{>} \surd\!\!\!\surd$;
- if $\alpha_i \in \mathcal{A}$, then $s_i = \alpha_i$ and $t_{i-1} \xrightarrow{\alpha_i}_{>} t_i$.

Finally, we write $|s_1 \ldots s_h|$ for the *length* of $s_1 \ldots s_h$.

Proposition 2. *Let t be a process term, σ a closed substitution, $n \in \mathbb{N}$ and p a process. If $\sigma(t) \rightarrow^n_{>} p$ then:*

1. *there exist a process term t' and a string $w \in (\mathcal{A} \cup \mathcal{V})^*$ and $s_1 \ldots s_h \in \mathcal{A}^*$ such that $t \xrightarrow{s_1 \ldots s_h}_{>,w} t'$, $\sigma(t') = p$ and $|s_1 \ldots s_h| = n$;*
2. *or $t \xrightarrow{s_1 \ldots s_h}_{>,w} t'$ for some $w \in (\mathcal{A} \cup \mathcal{V})^*$ and $s_1 \ldots s_h$ such that $|s_1 \ldots s_h| = k < n$, and there are a variable x, a natural number $l \in \mathbb{N}$ and a process q, such that $t' \xrightarrow{x_s}_{>} \odot^l(x_d)$, $\sigma(x) \rightarrow^{n-k}_{>} q$ and $\odot^l(q) = p$.*

The following result allows us to establish whether the behavior of two bisimilar process terms is determined by the same variable. Moreover, it guarantees that such a variable is enabled in one term if and only if it is enabled in the other one.

Theorem 1. *Assume that $\mathcal{A}$ contains at least two actions, a and b. Let x be a variable. Consider two process terms t and u such that $\mathcal{A}^*(t) \subseteq \{a\}$ and $t \underline{\leftrightarrow}_* u$. Whenever there is t' such that $t \rightarrow^k t'$, for some $k \in \mathbb{N}$, and $x \triangleleft_l t'$, for some $l \in \mathbb{N}$, then there is u' such that $u \rightarrow^k u'$ and $x \triangleleft_m u'$ for some $m \in \mathbb{N}$. Moreover, $l = 0$ if and only if $m = 0$.*

4 Determinate processes

As outlined in Section 2, $\underline{\leftrightarrow}_*$ cannot be defined conidINDUCTIVELY, contrary to the other bisimulation relations. However, in this section we identify a class of processes for which the coinductive reasoning on $\underline{\leftrightarrow}_*$ can be at least partially recovered, and which will be useful later on.

Definition 4 (Determinate process). *Let p be a process. We say that p is* uniformly determinate *if $|\mathcal{A}(p)| = 1$, and for all processes p_1 and p_2 such that $p \rightarrow p_1$ and $p \rightarrow p_2$, we have $\mathrm{norm}\,(p_1) = \mathrm{norm}\,(p_2) = 1$ and $p_1 \underline{\leftrightarrow}_* p_2$. Then, for each $k \in \mathbb{N}$, we say that p is* uniformly k-determinate *if whenever $p \rightarrow^k p'$ then p' is uniformly determinate.*

Thus, a process is uniformly k-determinate if whenever it takes k steps, it ends up in a process that only has one available action, and in which all immediate successors have norm 1 and are order-insensitive bisimilar. This notion of uniformly k-determinacy is preserved by order-insensitive bisimilarity.

Lemma 5. *If $p \underline{\leftrightarrow}_* q$ and p is uniformly k-determinate for all $1 \leq k < \mathrm{depth}(p)$, then so is q.*

The next Proposition shows that if p and q are order-insensitive bisimilar as well as uniformly k-determinate for all k less than some n, then every sequence of n transitions that p can do can be matched by q such that p and q end up in processes that are again order-insensitive bisimilar.

Proposition 3. *Let p and q be two processes such that $p \underline{\leftrightarrow}_* q$ and there is an $n \in \mathbb{N}$ such that p and q are uniformly k-determinate for all $k < n$. Suppose that $p \to^n p'$ for some p'. Then there is a process q' such that $q \to^n q'$ and $p' \underline{\leftrightarrow}_* q'$.*

Note that Proposition 3 can also be proved for a weaker notion of determinacy, but we need the current, stronger definition for the subsequent development.

5 Order-insensitive bisimilarity is not finitely based over BPA$_\Theta$

This section is devoted to our main result, namely that order-insensitive bisimilarity has no finite, ground-complete axiomatization in the setting of BPA$_\Theta$. Our proof strategy will be the following:

1. We define the property of *uniform Θ-n-dependency*: a process has this property if, through a sequence of n transitions among processes of norm 1, it can reach a process which is Θ-dependent in the sense of [3], namely its set of initial actions varies with the considered priority order.
2. We prove that by choosing n *large enough*, given a process p which is uniformly Θ-n-dependent and a finite set of axioms E, if $E \vdash p \approx q$, then q must be uniformly Θ-n-dependent.
3. We provide an infinite family of sound equations in which one side is uniformly Θ-n-dependent, but the other one is not. In light of item 2, this means that such a family of equations cannot be derived by a finite set of axioms and it must be included in the axiomatization, which is therefore infinite.

We actually start by defining the family of equations. To this end, we make use of the following processes, which are defined for each $n \in \mathbb{N}$ as

$$P_n = A_n(a) + A_n(b) + A_n(a + b),$$

where $A_0(p) = p$ and $A_n(p) = a \cdot A_{n-1}(p) + a$. Intuitively, the process P_n must at the top level decide whether it will end up in a, b, or $a + b$ after n steps. After making this choice, it can take up to n a-transitions, and at each step it can choose whether to terminate or to continue. The possibility of termination at each step is crucial, since it means that the process cannot be written just with sequential composition modulo bisimilarity.

The family of equations that we consider is then

$$\{P_n + A_n(\Theta(a+b)) \approx P_n \mid n \in \mathbb{N}\}. \tag{1}$$

Proposition 4. *For every $n \in \mathbb{N}$, the equation $P_n + A_n(\Theta(a+b)) \approx P_n$ is sound.*

Next we formalize the uniform Θ-n-dependency property. As previously outlined, this is based on the notion of Θ-*dependent* process from [3].

Definition 5 (Θ-dependent process, [3]). *A process p is* Θ-dependent *if there exist priority orders $>_1$ and $>_2$ such that $\mathcal{A}_{>_1}(p) \neq \mathcal{A}_{>_2}(p)$.*

Intuitively, a process is Θ-dependent if its possible behaviour depends on the choice of priority order. For example, $\Theta(a+b)$ is Θ-dependent, since we can find a priority order that only allows it to make a a-transition, and another priority order that only allows it to make a b-transition. On the other hand, $\Theta(a)$ is not Θ-dependent, since no matter what priority order we choose, it can only do a a-transition.

Uniform Θ-n-dependency is an extension of Θ-dependency from [3], in that it requires first that it is possible to take a sequence of n transitions and end up in a process that is Θ-dependent, and furthermore it mandates that at each step along the way, the process has a norm of 1.

Definition 6. *A process p is* uniformly Θ-n-dependent *if there are processes $p_1, \ldots, p_n$ such that $p = p_0 \to p_1 \to \cdots \to p_n$, the process p_n is Θ-dependent, and for all $0 \leq k < n$ we have* $\mathrm{norm}\,(p_k) = 1$.

We remark that the processes on the right-hand side of equations in (1) do not enjoy this property, whereas those on the left-hand side are uniformly Θ-n-dependent.

The following proposition tells us that Θ-n-dependency is preserved by closed instantiations of sound equations whose depth is smaller than n and that satisfy some determinacy constraints.

Proposition 5. *Let σ be a closed substitution and let t and u be process terms such that $t \underline{\leftrightarrow}_* u$ and $\mathcal{A}^*(t) = \{a\}$. Assume a natural number $n \in \mathbb{N}$ such that $n >$ $\max\{\mathrm{depth}\,(t)\,, \mathrm{depth}\,(u)\}$ and $\sigma(t)$ is uniformly k-determinate for all $1 \leq k \leq n-1$. If $\sigma(t)$ is uniformly Θ-n-dependent, then so is $\sigma(u)$.*

The final ingredient that we need for our main result is a way of relating arbitrary processes to the processes of the form P_n.

Definition 7 (Summand, [3]). *We say that p* is a summand of *q, denoted by $p \sqsubseteq_* q$, if there exists a process r such that $p + r \underline{\leftrightarrow}_* q$.*

The point of this definition is that any process p such that $p \sqsubseteq_* P_n$ must be of a specific form that inherits many of the features of P_n. In particular, such a process must be k-determinate for all k less than n.

Lemma 6. *Let p be a process and assume $p \sqsubseteq_* P_n$ for some $n \in \mathbb{N}$. Then p is uniformly k-determinate for all $1 \leq k < n$.*

We now arrive at our main theorem, which states that for n large enough, if p and q are summands of P_n, that can be proved equivalent from a finite set of sound equations, and p is Θ-n-dependent, then q must also be Θ-n-dependent.

Theorem 2. *Assume that $\mathcal{A}$ contains at least two actions. Let E be a set of sound process equations of depth less than n, and let p and q be closed processes such that $p, q \sqsubseteq_* P_n$ and $E \vdash p \approx q$. If p is uniformly Θ-n-dependent, then q is also uniformly Θ-n-dependent.*

As the left-hand side of the equations in (1) is Θ-n-dependent while the right-hand side is not, we can directly conclude that for each n, the nth instance of the family of equations in (1) cannot be proved using the infinite collection of all sound equations whose depth is smaller than n.

Corollary 1. *If $\mathcal{A}$ has at least two actions, then there is no finite set of sound equations E such that all sound process equations can be derived from E.*

6 Conclusions

In this work we have studied the finite axiomatizability of the equational theory of order-insensitive bisimilarity over the language BPA enriched with the priority operator Θ. As previous similar work suggested, also in this setting, the collection of sound, closed equations is not finitely based in the presence of at least two actions, despite the fact that the sequential composition operator allows one to write more complex axioms than action prefixing. We proved this negative result using an infinite family of closed equations suggested in [3] and showing that no set of sound equations of bounded depth can derive them all.

Finding an infinite (ground-)complete equational axiomatization of order-insensitive bisimilarity is an interesting avenue for future research. We also plan to study whether one can give a finite basis for the equational theory of order-insensitive bisimilarity using auxiliary operators, as has been done for bisimilarity for a variety of process algebras in the past [1,5,8]. In that study, we would be interested in developing a *minimal* set of auxiliary operators and in investigating their expressiveness.

Finally, we would like to investigate some algorithms, and their complexity, for checking order-insensitive bisimilarity of (loop-free) finite labelled transition systems. It is known that bisimilarity over such systems is P-complete [6], and, moreover, using the Paige-Tarjan algorithm [17] each $\underline{\leftrightarrow}_>$ can be checked in $O(m \log n)$, where m is the number of transitions and n is the number of states. A naive algorithm for $\underline{\leftrightarrow}_*$ would then check $\underline{\leftrightarrow}_>$ for all the possible partial orders $>$ over $\mathcal{A}$. Assuming that $|\mathcal{A}| = k$, there are $2^{k^2/4+3k/4+O(\log k)}$ possible partial orders (see [14] for the result on the number of posets over sets with k elements). Clearly, from these results we can obtain an upper bound on the complexity of $\underline{\leftrightarrow}_*$. It would be then interesting to look for the lower bounds on the complexity of deciding order-insensitive bisimilarity and for heuristics that might lead to algorithms that improve on the naive one, as for some orders $>_1, >_2$ the computations made to check $\underline{\leftrightarrow}_{>_1}$ could be partially re-used to check $\underline{\leftrightarrow}_{>_2}$.

References

1. Aceto, L., Bloom, B., Vaandrager, F.W.: Turning SOS rules into equations. Information and Computation **111**(1), 1–52 (1994), https://doi.org/10.1006/inco.1994.1040
2. Aceto, L., Chen, T., Fokkink, W., Ingólfsdóttir, A.: On the axiomatizability of priority. In: ICALP (2). Lecture Notes in Computer Science, vol. 4052, pp. 480–491. Springer (2006)
3. Aceto, L., Chen, T., Ingólfsdóttir, A., Luttik, B., van de Pol, J.: On the axiomatizability of priority II. Theor. Comput. Sci. **412**(28), 3035–3044 (2011). https://doi.org/10.1016/j.tcs.2011.02.033
4. Aceto, L., Fokkink, W., Ingólfsdóttir, A., Nain, S.: Bisimilarity is not finitely based over BPA with interrupt. Theor. Comput. Sci. **366**(1-2), 60–81 (2006). https://doi.org/10.1016/j.tcs.2006.07.003
5. Baeten, J.C., Bergstra, J.A., Klop, J.W.: Syntax and defining equations for an interrupt mechanism in process algebra. Fundamenta Informaticae **IX**(2), 127–168 (1986)
6. Balcázar, J.L., Gabarró, J., Santha, M.: Deciding bisimilarity is P-complete. Formal Asp. Comput. **4**(6A), 638–648 (1992)
7. Bergstra, J.: Put and get, primitives for synchronous unreliable message passing. Logic Group Preprint Series Nr. 3, CIF, State University of Utrecht (1985)
8. Bergstra, J.A., Klop, J.W.: Process algebra for synchronous communication. Information and Control **60**(1-3), 109–137 (1984). https://doi.org/10.1016/S0019-9958(84)80025-X
9. Bloom, B., Istrail, S., Meyer, A.R.: Bisimulation can't be traced. J. ACM **42**(1), 232–268 (1995)
10. Cleaveland, R., Lüttgen, G., Natarajan, V.: Chapter 12 - priority in process algebra. In: Bergstra, J., Ponse, A., Smolka, S. (eds.) Handbook of Process Algebra, pp. 711 – 765. Elsevier Science, Amsterdam (2001). https://doi.org/https://doi.org/10.1016/B978-044482830-9/50030-8
11. van Glabbeek, R.J.: The meaning of negative premises in transition system specifications II. In: Proceedings of ICALP'96. pp. 502–513. Lecture Notes in Computer Science (1996)
12. Hoare, C.A.R.: Communicating sequential processes. Commun. ACM **21**(8), 666–677 (1978). https://doi.org/10.1145/359576.359585
13. Hoare, C.A.R.: Communicating Sequential Processes. Prentice-Hall (1985)
14. Kleitman, D.J., Rothschild, B.L.: Asymptotic enumeration of partial orders on a finite set. Trans. Amer. Math. Soc. **205**, 205–220 (1975). https://doi.org/10.2307/1997200
15. Milner, R.: A Calculus of Communicating Systems, Lecture Notes in Computer Science, vol. 92 (1980). https://doi.org/10.1007/3-540-10235-3
16. Milner, R.: Communication and concurrency. PHI Series in computer science, Prentice Hall (1989)
17. Paige, R., Tarjan, R.E.: Three partition refinement algorithms. SIAM J. Comput. **16**(6), 973–989 (1987). https://doi.org/10.1137/0216062
18. Plotkin, G.D.: A structural approach to operational semantics. Report DAIMI FN-19, Aarhus University (1981)
19. Vrancken, J.L.M.: The algebra of communicating processes with empty process. Theor. Comput. Sci. **177**(2), 287–328 (1997). https://doi.org/10.1016/S0304-3975(96)00250-2

A Depth and norm

Since in our setting the length of sequences of enables transitions depends on the considered priority order, we define the depth and the norm of a process with respect to the empty order. The reason for this choice is twofold. Firstly, we notice that the depth defined with respect to the empty order is an upper bound for the depths defined with respect to any other priority order. Since for our purposes we will need to consider upper bounds for the depth of processes, and not the exact value of their depths, it is reasonable to consider directly the greatest of the depths. Notice that the norm defined with respect to the empty order is, dually, a lower bound for the norms defined with respect to the other priority orders. Secondly, this choice allows us to give alternative formulations of both measures by induction on the structure of processes.

Definition 8 (Depth and norm of processes). *The* depth *of a process is defined inductively on its structure by*

- $\mathrm{depth}\,(a) = 1$*;*
- $\mathrm{depth}\,(p_1 \cdot p_2) = \mathrm{depth}\,(p_1) + \mathrm{depth}\,(p_2)$*;*
- $\mathrm{depth}\,(p_1 + p_2) = \max\{\mathrm{depth}\,(p_1)\,, \mathrm{depth}\,(p_2)\}$*;*
- $\mathrm{depth}\,(\Theta(p)) = \mathrm{depth}\,(p)$.

Similarly, the norm *of process is defined inductively on its structure by*

- $\mathrm{norm}\,(a) = 1$*;*
- $\mathrm{norm}\,(p_1 \cdot p_2) = \mathrm{norm}\,(p_1) + \mathrm{norm}\,(p_2)$*;*
- $\mathrm{norm}\,(p_1 + p_2) = \min\{\mathrm{norm}\,(p_1)\,, \mathrm{norm}\,(p_2)\}$*;*
- $\mathrm{norm}\,(\Theta(p)) = \mathrm{norm}\,(p)$.

Both notions can be extended to process terms *by adding, respectively, the value of the depth and norm of a variable which are defined as* $\mathrm{depth}\,(x) = 1$ *and* $\mathrm{norm}\,(x) = 1$.

We remark that although variables cannot perform any transition, as one can easily see from the inference rules in Table 1, their depth, and norm, are set to 1, since the minimal closed instance of a variable with respect to these measures is as a constant in $\mathcal{A}$.

B Proofs of results in Section 3

B.1 Proof of Lemma 2

Proof of Lemma 2.

1. We proceed by induction over the derivation of the predicate $t \xrightarrow{x}_{>} \surd\!\!\surd$.
   - Base case: $t = x$ and $t \xrightarrow{x}_{>} \surd\!\!\surd$ is derived by rule (a_2) in Table 3. Hence the proof directly follows by $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$.
   - Inductive step: $t = t_1 + t_2$ and $t \xrightarrow{x}_{>} \surd\!\!\surd$ is derived by either rule (a_8) in Table 3, and thus by $t_1 \xrightarrow{x}_{>} \surd\!\!\surd$, or its symmetric version on t_2. Assume wlog. that rule (a_8) in Table 3 was applied. Then by induction $t_1 \xrightarrow{x}_{>} \surd\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ imply $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\surd$. Hence, the premise of rule (r_4) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\surd$.

- Inductive step: $t = \Theta(u)$ and $t \xrightarrow{x}_{>} \surd\!\!\!\surd$ is derived by rule (a_{11}) in Table 3, and thus by $u \xrightarrow{x}_{>} \surd\!\!\!\surd$. By induction $u \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ imply $\sigma(u) \xrightarrow{a}_{>} \surd\!\!\!\surd$. Since by the hypothesis action a has maximal priority with respect to $>$, the premises of rule (r_8) in Table 1 are satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\!\surd$.

2. We proceed by induction over the derivation of the auxiliary transition $t \xrightarrow{x}_{>} t'$.
   - Base case: $t = t_1 \cdot t_2$ and $t \xrightarrow{x}_{>} t'$ is derived by rule (a_5) in Table 3, namely $t_1 \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $t' = t_2$. By Lemma 2.1 we have that $t_1 \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\!\surd$. Hence, the premise of rule (r_2) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(t_2)$.
   - Inductive step: $t = t_1 \cdot t_2$ and $t \xrightarrow{x}_{>} t'$ is derived by rule (a_4) in Table 3, namely $t_1 \xrightarrow{x}_{>} t_1'$ and $t' = t_1' \cdot t_2$. By induction we have that $t_1 \xrightarrow{x}_{>} t_1'$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \sigma(t_1')$. Hence, the premise of rule (r_3) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(t_1' \cdot t_2)$.
   - Inductive step: $t = t_1 + t_2$ and $t \xrightarrow{x}_{>} t'$ is derived either by rule (a_7) in Table 3, namely $t_1 \xrightarrow{x}_{>} t_1'$ and $t' = t_1'$, or by its symmetric version for t_2. Assume wlog. that rule (a_7) was applied. By induction we have that $t_1 \xrightarrow{x}_{>} t_1'$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \sigma(t_1')$. Hence, the premise of rule (r_6) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(t_1')$.
   - Inductive step: $t = \Theta(u)$ and $t \xrightarrow{x}_{>} t'$ is derived by rule (a_{10}) in Table 3, namely $t_1 \xrightarrow{x}_{>} t_1'$ and $t' = \Theta(t_1')$. By induction we have that $t_1 \xrightarrow{x}_{>} t_1'$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ imply that $\sigma(t_1) \xrightarrow{a}_{>} \sigma(t_1')$. Since by the hypothesis action a has maximal priority with respect to $>$, the premise of rule (r_9) in Table 1 is satisfied and we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma(\Theta(t_1'))$.
3. We proceed by induction over the derivation of the auxiliary transition $t \xrightarrow{x_s}_{>} c$.
   - Base case: $t = x$ and $t \xrightarrow{x_s}_{>} c$ is derived by rule (a_1) in Table 3, namely $c = x_d$. Hence the proof follows directly by $\sigma(x) \xrightarrow{a}_{>} p$.
   - Inductive step: $t = t_1 \cdot t_2$ and $t \xrightarrow{x_s}_{>} c$ is derived by rule (a_3) in Table 3, namely $t_1 \xrightarrow{x_s}_{>} c'$ and $c = c' \cdot t_2$. By induction we have that $t_1 \xrightarrow{x_s}_{>} c'$ and $\sigma(x) \xrightarrow{a}_{>} p$ imply $\sigma(t_1) \xrightarrow{a}_{>} p'$ for $p' = \sigma[x_d \mapsto p](c')$. Hence, by rule (r_3) in Table 1 we can infer that $\sigma(t) \xrightarrow{a}_{>} p' \cdot \sigma(t_2)$, with $p' \cdot \sigma(t_2) = \sigma[x_d \mapsto p](c' \cdot t_2)$.
   - Inductive step: $t = t_1 + t_2$ and $t \xrightarrow{x_s}_{>} c$ is derived either by rule (a_6) in Table 3, namely $t_1 \xrightarrow{x_s}_{>} c$, or by its symmetric version for t_2. Assume wlog. that (a_6) was applied. By induction we have that $t_1 \xrightarrow{x_s}_{>} c$ and $\sigma(x) \xrightarrow{a}_{>} p$ imply $\sigma(t_1) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$. Hence, by rule (r_6) in Table 1 we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$.
   - Inductive step: $t = \Theta(u)$ and $t \xrightarrow{x_s}_{>} \Theta(c)$ is derived by rule (a_9) in Table 3, namely $u \xrightarrow{x_s}_{>} c$. By induction we have that $u \xrightarrow{x_s}_{>} c$ and $\sigma(x) \xrightarrow{a}_{>} p$ imply $\sigma(u) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](c)$. Since by the hypothesis action a has maximal priority with respect to $>$, by rule (r_9) in Table 1 we can infer that $\sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p](\Theta(c))$.

□

B.2 Proof of Lemma 3

Before proceeding to the proof, the provide an auxiliary technical Lemma, that will simplify our reasoning.

Lemma 7. *Let $a \in \mathcal{A}$ be maximal with respect to $>$, and let σ be a closed substitution. Consider a configuration c, and processes p, p' s.t. $p \xrightarrow{a}_{>} p'$. If c contains an occurrence of x_d, then $\sigma[x_d \mapsto p](c) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c)$.*

Proof. We proceed by structural induction on c.

- Base case $c = t$: since c does not contain an occurrence of x_d, the lemma is vacuously true.
- Base case $c = x_d$: clearly, $\sigma[x_d \mapsto p](c) = p \xrightarrow{a}_{>} p' = \sigma[x_d \mapsto p'](c)$.
- Inductive step $c = c' \cdot t$: by induction over c' we obtain $\sigma[x_d \mapsto p](c') \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c')$. An application of rule (r_3) in Table 1 therefore gives

  $$\sigma[x_d \mapsto p](c) = \sigma[x_d \mapsto p](c') \cdot \sigma(t) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c') \cdot \sigma(t) = \sigma[x_d \mapsto p'](c).$$

- Inductive step $c = \Theta(c')$: by induction over c' we have $\sigma[x_d \mapsto p](c') \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](c')$. Since moreover a is maximal with respect to $>$, by applying rule (r_9) in Table 1 we obtain

  $$\sigma[x_d \mapsto p](c) = \sigma[x_d \mapsto p](\Theta(c')) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](\Theta(c')) = \sigma[x_d \mapsto p'](c).$$

□

Proof of Lemma 3. Note first of all that since $t \xrightarrow{x_s}_{>} c$, then c must contain an occurrence of x_d.

We proceed by induction on the derivation of $t \xrightarrow{x_s}_{>} c$, and for each case, we prove the statement by induction on n. However, since the base case of $n = 1$ is given by Lemma 2.3, we omit the details here. Furthermore, in each case we will use that fact that $\sigma(x) \to_{>}^{n} p$ implies that $\sigma(x) \to_{>}^{n-1} p' \to_{>} p$ for some process p'.

Rule (a_1): In this case we have $t = x$ and $c = x_d$. By induction hypothesis we get

$$\sigma(t) = \sigma(x) \to_{>}^{n-1} \sigma[x_d \mapsto p'](x_d) = p',$$

and we know that $p' \to_{>} p = \sigma[x_d \mapsto p](c)$, so we conclude that $\sigma(t) \to_{>}^{n} \sigma[x_d \mapsto p](c)$.

Rule (a_3): In this case we have $t = t_1 \cdot t_2$, $t_1 \xrightarrow{x_s}_{>} c'$, and $c = c' \cdot t_2$. By the induction hypothesis we get $\sigma(t_1) \to_{>}^{n-1} \sigma[x_d \mapsto p'](c')$, which, by rule (r_3), gives

$$\sigma(t) = \sigma(t_1) \cdot \sigma(t_2) \to_{>}^{n-1} \sigma[x_d \mapsto p'](c') \cdot \sigma(t_2) = \sigma[x_d \mapsto p'](c).$$

Since $p' \to_{>} p$, Lemma 7 gives $\sigma[x_d \mapsto p'](c) \to_{>} \sigma[x_d \mapsto p](c)$, so we conclude $\sigma(t) \to_{>}^{n} \sigma[x_d \mapsto p](c)$.

Rule (a_6): In this case we have $t = t_1 + t_2$ and $t_1 \xrightarrow{x_s}_> c$. The induction hypothesis gives $\sigma(t_1) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](c)$, so rule ($r_6$) and Lemma 7 together give

$$\sigma(t) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](c) \rightarrow_> \sigma[x_d \mapsto p](c).$$

A similar argument using rule (r_7) establishes the symmetric case.

Rule (a_9): In this case we have $t = \Theta(t')$, $t' \xrightarrow{x_s}_> c'$, and $c = \Theta(c')$. By the induction hypothesis we get $\sigma(t_1) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](c')$. Rule ($r_9$) and Lemma 7 then give

$$\sigma(t) \rightarrow_>^{n-1} \sigma[x_d \mapsto p'](\Theta(c')) = \sigma[x_d \mapsto p'](c) \rightarrow_> \sigma[x_d \mapsto p](c).$$

□

B.3 Proof of Lemma 4

Proof of Lemma 4. ($\Longrightarrow$) We proceed by structural induction on t in $x \lhd_\ell t$.

Case 1: We have $x \lhd_0 x$, so $x = t$, and hence rule (a_1) gives $t \xrightarrow{x_s}_> x_d = c_0$.

Case 2: We have $t = t_1 + t_2$ and either $x \lhd_\ell t_1$ or $x \lhd_\ell t_2$. If $x \lhd_\ell t_1$, then by induction hypothesis we get $t_1 \xrightarrow{x_s}_> c_\ell$, so rule ($a_6$) gives $t \xrightarrow{x_s}_> c_\ell$. If $x \lhd_\ell t_2$, we get the same by result by the symmetric version of (a_6).

Case 3: We have $t = t_1 \cdot t_2$ and $x \lhd_\ell t_1$, so by induction hypothesis we get $t_1 \xrightarrow{x_s}_> c'_\ell$. Rule ($a_3$) then gives $t \xrightarrow{x_s}_> c'_\ell \cdot t_2 = c_\ell$, which is of the correct form.

Case 4: We have $t = \Theta(t')$ and $x \lhd_{\ell-1} t'$. By induction hypothesis we get $t' \xrightarrow{x_s}_> c'_{\ell-1}$. Rule ($a_9$) gives $t \xrightarrow{x_s}_> \Theta(c'_{\ell-1}) = c_\ell$, which is of the correct form.

($\Longleftarrow$) The proof is by induction on the derivation of $t \xrightarrow{x_s}_> c_\ell$.

Rule (a_1): In this case we have $\ell = 0$, $t = x$, and $x \lhd_0 t$.

Rule (a_3): We have $t = t_1 \cdot t_2$ with $t_1 \xrightarrow{x_s}_> c'_\ell$. By induction hypothesis, this gives $x \lhd_\ell t_1$, which implies $x \lhd_\ell t_1 \cdot t_2 = t$.

Rule (a_6): We have $t = t_1 + t_2$ with $t_1 \xrightarrow{x_s}_> c_\ell$. The induction hypothesis then gives $x \lhd_\ell t_1$, which implies $x \lhd_\ell t_1 + t_2 = t$. The same argument holds for the symmetric version of (a_6).

Rule (a_9): We have $t = \Theta(t')$ with $t' \xrightarrow{x_s}_> c_{\ell-1}$. By induction hypothesis, this gives $x \lhd_{\ell-1} t'$, which implies $x \lhd_\ell \Theta(t') = t$.

□

B.4 Proof of Proposition 1

Proof of Proposition 1.

1. We proceed by induction over the derivation of $\sigma(t) \xrightarrow{a}_> \surd\!\!\surd$.
   - Base cases: $t = a$ and $t = x$. The proof for the former case follows directly by rule (r_1) in Table 1 and the latter directly by rule (a_2) in Table 3.
   - Inductive step $t = t_1 + t_2$ and $\sigma(t) \xrightarrow{a}_> \surd\!\!\surd$ is derived either by rule (r_4) in Table 1, and thus by $\sigma(t_1) \xrightarrow{a}_> \surd\!\!\surd$, or by rule ($r_5$) in Table 1, and thus by $\sigma(t_2) \xrightarrow{a}_> \surd\!\!\surd$. Assume wlog. that rule (r_4) was applied. By induction over $\sigma(t_1) \xrightarrow{a}_> \surd\!\!\surd$ we can distinguish two cases:

- $t_1 \xrightarrow{a}_{>} \surd\!\!\!\surd$. Then by rule (r_4) in Table 1 we derive that $t \xrightarrow{a}_{>} \surd\!\!\!\surd$.
- There is a variable x s.t. $t_1 \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$. Hence, by applying rule (a_8) in Table 3 we derive that, for the same variable x, $t \xrightarrow{x}_{>} \surd\!\!\!\surd$.

– Inductive step: $t = \Theta(u)$ and $\sigma(t) \xrightarrow{a}_{>} \surd\!\!\!\surd$ is derived by rule (r_8) in Table 1. This implies that $\sigma(u) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(u) \stackrel{b}{\not\to}_{>}$ for all $b > a$. By induction over $\sigma(u) \xrightarrow{a}_{>} \surd\!\!\!\surd$ we can distinguish two cases:
  - $u \xrightarrow{a}_{>} \surd\!\!\!\surd$. Since moreover from $\sigma(u) \stackrel{b}{\not\to}_{>}$ for all $b > a$ we can infer that $u \stackrel{b}{\not\to}_{>}$ for all such b, the premises of rule (r_8) in Table 1 are satisfied and we can derive that $t \xrightarrow{a}_{>} \surd\!\!\!\surd$.
  - There is a variable x s.t. $u \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$. By applying rule (a_{11}) in Table 3 we derive that, for the same variable, $t \xrightarrow{x}_{>} \surd\!\!\!\surd$.

2. We proceed by induction over the derivation of $\sigma(t) \xrightarrow{a}_{>} p$.
   – Base case: $t = x$. Then case (2c) is satisfied directly by rule (a_1) in Table 3.
   – Inductive step: $t = t_1 \cdot t_2$. We can distinguish two cases:
   - $\sigma(t) \xrightarrow{a}_{>} p$ is derived by rule (r_2) in Table 1, namely by $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $p = \sigma(t_2)$. From $\sigma(t_1) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and Proposition 1.1 we get that either $t_1 \xrightarrow{a}_{>} \surd\!\!\!\surd$ or there is a variable x s.t. $t_1 \xrightarrow{x}_{>} \surd\!\!\!\surd$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$. In the former case we can apply rule (r_2) in Table 1 and obtain $t \xrightarrow{a}_{>} t_2$ with $\sigma(t_2) = p$, thus case (2a) is satisfied. In the latter case we can apply rule (a_5) in Table 3 and obtain $t \xrightarrow{x}_{>} t_2$ which together with $\sigma(t_2) = p$ and $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ satisfies case (2b).
   - $\sigma(t) \xrightarrow{a}_{>} p$ is derived by rule (r_3) in Table 1, namely by $\sigma(t_1) \xrightarrow{a}_{>} p_1$ with $p_1 = q \cdot \sigma(t_2)$. By induction over $\sigma(t_1) \xrightarrow{a}_{>} p_1$ we can distinguish three cases:
     * Case (2a) applies so that there is a process term t_1' s.t. $t_1 \xrightarrow{a}_{>} t_1'$ and $\sigma(t_1') = p_1$. Then, by rule (r_3) in Table 1 we infer that $t \xrightarrow{a}_{>} t_1' \cdot t_2$ with $\sigma(t_1') \cdot \sigma(t_2) = p$, and thus case (2a) is also satisfied by t.
     * Case (2b) applies so that there is a process term t_1' and a variable x s.t. $t_1 \xrightarrow{x}_{>} t_1'$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(t_1') = p_1$. Then, by rule (a_4) in Table 3 we infer that $t \xrightarrow{x}_{>} t_1' \cdot t_2$ with $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(t_1') \cdot \sigma(t_2) = p$, and thus case (2b) is also satisfied by t.
     * Case (2c) applies so that there are a variable x, a natural $l \in \mathbb{N}$ and a process s s.t. $t_1 \xrightarrow{x_s}_{>} \odot^l(x_d)$, $\sigma(x) \xrightarrow{a}_{>} q$ and $\odot^l(q) = p_1$. Then, by rule (a_3) in Table 3 we infer that $t \xrightarrow{x_s}_{>} \odot^l(x_d) \cdot t_2$. Hence case (2c) is also satisfied by t with respect to the configuration $\odot^l(x_d) \cdot t_2$, the variable x, the natural $l \in \mathbb{N}$ and the process q for which $\odot^l(q) \cdot \sigma(t_2) = p$.

   – Inductive step: $t = t_1 + t_2$ and $\sigma(t) \xrightarrow{a}_{>} p$ is derived by the same transition performed either by $\sigma(t_1)$ or $\sigma(t_2)$, namely by applying either rule (r_6) or rule (r_7) in Table 1. Since induction applies to such a move taken by $\sigma(t_i)$ and in all the rules for nondeterministic choice in Tables 1 and 3 the moves of t_i are mimicked exactly by t, we can infer that each of the three cases of Proposition 1.2 holds for t whenever it holds for t_i.

– Inductive step: $t = \Theta(u)$ and $\sigma(t) \xrightarrow{a}_{>} p$ is derived by applying rule (r_9) in Table 1. This implies that $\sigma(u) \xrightarrow{a}_{>} p_1$, with $\Theta(p_1) = p$, and $\sigma(u) \not\xrightarrow{b}_{>}$ for all $b > a$. By induction over $\sigma(u) \xrightarrow{a}_{>} p_1$ we can distinguish three cases:
  - Case (2a) applies so that there is a process term u' s.t. $u \xrightarrow{a}_{>} u'$ and $\sigma(u') = p_1$. Moreover, we can also infer that $u \not\xrightarrow{b}_{>}$ for all $b > a$ because $\sigma(u) \not\xrightarrow{b}_{>}$. Then, by rule (r_9) in Table 1 we infer that $t \xrightarrow{a}_{>} \Theta(u')$ with $\sigma(\Theta(u')) = p$, and thus case (2a) is also satisfied by t.
  - Case (2b) applies so that there is a process term u' and a variable x s.t. $u \xrightarrow{x}_{>} u'$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(u') = p_1$. Then, by rule (a_{10}) in Table 3 we infer that $t \xrightarrow{x}_{>} \Theta(u')$ with $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(\Theta(u')) = p$, and thus case (2b) is also satisfied by t.
  - Case (2c) applies so that there are a variable x, a natural $l \in \mathbb{N}$ and a process q s.t. $u \xrightarrow{x_s}_{>} \odot^l(x_d)$, $\sigma(x) \xrightarrow{a}_{>} q$ and $\odot^l(q) = p_1$. Then, by rule (a_9) in Table 3 we infer that $t \xrightarrow{x_s}_{>} \Theta(\odot^l(x_d)) = \odot^{l+1}(x_d)$. Hence case (2c) is also satisfied by t with respect to the variable x, the natural $l + 1$ and the process q for which $\odot^{l+1}(q) = p$.

□

B.5 Proof of Proposition 2

Before proceeding to the proof, we notice that by Lemma 7, if $p \xrightarrow{a}_{>} p'$ for some action a having (locally) maximal priority with respect to $>$, then the transition $\sigma[x_d \mapsto p](\odot^l(x_d)) \xrightarrow{a}_{>} \sigma[x_d \mapsto p'](\odot^l(x_d))$ is well defined. In this case, we abuse notation slightly and write directly $\odot^l(p) \xrightarrow{a}_{>} \odot^l(p')$.

Proof of Proposition 2. We proceed by induction over n.

– Base case $n = 1$. This directly follows by Proposition 1.2.
– Inductive step $n > 1$. $\sigma(t) \rightarrow^n_{>} p$ is equivalent to write $\sigma(t) \rightarrow_{>} p_1 \rightarrow^{n-1}_{>} p$, for some process p_1. We can assume wlog. that $\sigma(t) \xrightarrow{a}_{>} p_1$. Accordingly to Proposition 1.2, from $\sigma(t) \xrightarrow{a}_{>} p_1$ we can distinguish three cases:
  1. there is a process term t_1 s.t. $t \xrightarrow{a}_{>} t_1$ and $\sigma(t_1) = p_1$. Then by induction over $p_1 \rightarrow^{n-1}_{>} p$ we can distinguish two subcases:
     - there is $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$ with $t_1 \xrightarrow{s_1 \dots s_h}_{>, w_1} t'$ with $|s_1 \dots s_h| = n - 1$ and $\sigma(t') = p$. Then, the proof can be concluded by noticing that for the sequence $w = aw_1$ we get $t \xrightarrow{as_1 \dots s_h}_{>, w} t'$ with $|as_1 \dots s_h| = n$ and $\sigma(t') = p$.
     - there are $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$, a variable y, a natural $l \in \mathbb{N}$ and a process q, such that $t_1 \xrightarrow{s_1 \dots s_h}_{>, w_1} t'$ with $|s_1 \dots s_h| = k < n - 1$, $t' \xrightarrow{y_s}_{>} \odot^l(y_d)$, $\sigma(y) \rightarrow^{n-1-k}_{>} q$ and $\odot^l(q) = p$. Then, the proof can be concluded by noticing that for the sequence $w = aw_1$ we get $t \xrightarrow{as_1 \dots s_h}_{>, w} t'$ with $|as_1 \dots s_h| = k + 1 < n$ and y, l, q behave as before.
  2. there are a process term t_1 and a variable x s.t. $t \xrightarrow{x}_{>} t_1$, $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\!\surd$ and $\sigma(t_1) = p_1$. Then by induction over $p_1 \rightarrow^{n-1}_{>} p$ we can distinguish two subcases:

- there is $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$ with $t_1 \xrightarrow{s_1 \dots s_h}_{>} t'$ with $|s_1 \dots s_h| = n-1$ and $\sigma(t') = p$. Then, the proof can be concluded by noticing that for the sequence $w = xw_1$ we get $t \xrightarrow{as_1 \dots s_h}_{>,w} t'$ with $|as_1 \dots s_h| = n$, as $|a| = 1$, and $\sigma(t') = p$.
- there are $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$, a variable y, a natural $l \in \mathbb{N}$ and a process q, such that $t_1 \xrightarrow{s_1 \dots s_h}_{>,w_1} t'$ with $|s_1 \dots s_h| = k < n-1$, $t' \xrightarrow{y_s}_{>} \odot^l(y_d)$, $\sigma(y) \to_{>}^{n-1-k} q$ and $\odot^l(q) = p$. Then, the proof can be concluded by noticing that, since $\sigma(x) \xrightarrow{a}_{>} \surd\!\!\surd$ gives $|a| = 1$, for the sequence $w = xw_1$ we get $t \xrightarrow{as_1 \dots s_h}_{>,w} t'$ with $|as_1 \dots s_h| = k+1 < n$ and c, x, q behave as before.

3. there are a variable x, a natural $m \in \mathbb{N}$ and a process p' s.t. $t \xrightarrow{x_s}_{>} \odot^m(x_d)$, $\sigma(x) \xrightarrow{a}_{>} p'$. More precisely, Lemma 3 allows us to distinguish two cases:
   - $\sigma(x) \to_{>}^{h} q$ for some $h \geq n$. In this case the thesis follows by considering $w = \emptyset$ and the process q' s.t. $\sigma(x) \to_{>}^{n} q'$ and $\odot^l(q') = p$.
   - $\sigma(x) \to_{>}^{k-1} q \to_{>} \surd\!\!\surd$ for some $k < n$. Notice that this implies that there is some string s_x with $|s_x| = k$ of actions that have been performed by $\sigma(x)$. Due to the structure of $\odot^l(x_d)$ we can infer that there are a natural $m' \in \mathbb{N}$ and a process term $t_1 = \underbrace{\Theta(\cdots\Theta}_{m' \text{ times}}(t'' \odot t_{m'+1}) \odot t_{m'}) \dots) \odot u_1$ s.t. $\sigma(t) \to_{>}^{k} \sigma(t_1) = p_1$. Since then $p_1 \to_{>}^{n-k} p$, by induction we can distinguish two subcases:
     - ∗ there is $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$ with $t_1 \xrightarrow{s_1 \dots s_h}_{>,w_1} t'$ with $|s_1 \dots s_h| = n-k$ and $\sigma(t') = p$. Then, the proof can be concluded by noticing that for the sequence $w = x_s w_1$ we get $t \xrightarrow{s_x s_1 \dots s_h}_{>,w} t'$ with $|s_x s_1 \dots s_h| = n$, as $|s_x| = k$, and $\sigma(t') = p$.
     - ∗ there are $w_1 \in (\mathcal{A} \cup \mathcal{V})^*$, a variable y, a natural $l \in \mathbb{N}$ and a process q', such that $t_1 \xrightarrow{s_1 \dots s_h}_{>,w_1} t'$ with $|s_1 \dots s_h| = j < n-k$, $t' \xrightarrow{y_s}_{>} \odot^l(y_d)$, $\sigma(y) \to_{>}^{n-k-j} q'$ and $\odot^l(q') = p$. Then, the proof can be concluded by noticing that, as $|s_x| = k$, for the sequence $w = x_s w_1$ we get $t \xrightarrow{s_x s_1 \dots s_h}_{>,w} t'$ with $|s_x s_1 \dots s_h| = k+j < n$ and y, l, q' behave as before.

□

B.6 Proof of Theorem 1

Proof of Theorem 1. Let $n \in \mathbb{N}$ be larger than the depths of t and u, and assume a priority order $>$ over $\mathcal{A}$ with $b > a$, with $a > c$ for any other possible action $c \in \mathcal{A}$. We define the family of closed substitutions $\{\sigma_i\}_{i \in \mathbb{N}}$ inductively as follows:

$$\sigma_0(y) = \begin{cases} a + b & \text{if } y = x \\ a & \text{otherwise.} \end{cases}$$

$$\sigma_i(y) = \begin{cases} a \cdot (\sigma_{i-1}(y) + a) & \text{if } y = x \\ a & \text{otherwise.} \end{cases}$$

Let $\sigma = \sigma_n$. Suppose that $t \to^k t'$, for some $k \in \mathbb{N}$. Since $\mathcal{A}^*(t) = \{a\}$, and all variables but x are mapped into a process that can only execute a, we can infer that there are process terms $t_0, \ldots, t_k$ s.t. $t = t_0 \xrightarrow{a} \ldots \xrightarrow{a} t_k = t'$. Moreover, as in all such terms t_i there is no occurrence of b, a is maximal with respect to $>$ on them, and thus by Lemma 1 and an easy induction over k, we obtain that $\sigma(t_0) \xrightarrow{a}^k \sigma(t_k)$, namely $\sigma(t) \xrightarrow{a}_{>}^k \sigma(t')$. Suppose now that $x \lhd_l t'$, for some $l \in \mathbb{N}$. By Lemma 4, $x \lhd_l t'$ implies that $t' \xrightarrow{x_s}_{>} \odot^l(x_d)$. By the choice of σ and $\mathcal{A}^*(t) = \{a\}$, we have that $\sigma(x) \xrightarrow{a}_{>}^n a + b$. Therefore, by Lemma 3 we obtain that $\sigma(t') \xrightarrow{a}_{>}^n \odot^l(a + b)$. By combining the two sequences of transitions, we get $\sigma(t) \xrightarrow{a}_{>}^{k+n} \odot^l(a + b)$. By the hypothesis we have $t \underline{\leftrightarrow}_* u$, which in particular implies $t \underline{\leftrightarrow}_> u$ and thus $\sigma(t) \underline{\leftrightarrow}_> \sigma(u)$. As $\underline{\leftrightarrow}_>$ is a bisimulation, we can infer that $\sigma(u) \xrightarrow{a}_{>}^{k+n} p$ for some process p with $\odot^l(a+b) \underline{\leftrightarrow}_> p$. As n is larger than the depth of u, by Proposition 2 there exist a process term u', a string w with strings $s_1, \ldots, s_h \in \{a\}^*$, a variable y, a natural number m and a process q such that $u \xrightarrow{s_1 \ldots s_h}_{>,w} u'$, $|s_1 \ldots s_h| = j < n$, $u' \xrightarrow{x_s}_{>} \odot^m(y_d)$, $\sigma(y) \to_{>}^{k+n-j} q$ and $p = \odot^m(q)$. Therefore: (i) by $k + n - j > 0$; (ii) by the choice of $>$ (which gives that the only possible transition enabled for $\odot^l(a + b)$ is a b-labeled move); (iii) by the choice of σ; (iv) by $\odot^l(a + b) \underline{\leftrightarrow}_> \odot^m(q)$; we can conclude that $y = x$, $j = k$ and $q = a + b$. Moreover, from $\odot^l(a + b) \underline{\leftrightarrow}_> \odot^m(a + b)$ and the choice of $>$, we obtain that $l = 0$ iff $m = 0$. □

C Proofs of results in Section 4

C.1 Proof of Lemma 5

Proof of Lemma 5. The proof is by induction on k. Note that $p \underline{\leftrightarrow}_* q$ in particular implies $p \underline{\leftrightarrow} q$.

Base case: If $k = 1$, assume that q is not uniformly 1-determinate. This means that either $|\mathcal{A}(q)| > 1$ or there exist q_1 and q_2 such that $q \to q_1$ and $q \to q_2$ but $q_1 \not\underline{\leftrightarrow}_\star q_2$, or $\mathrm{norm}\,(q_1) \neq 1$, or $\mathrm{norm}\,(q_2) \neq 1$.

If $|\mathcal{A}(q)| > 1$, then there are $a, b \in \mathcal{A}$ with $a \neq b$ such that $q \xrightarrow{a} q_a$ and $q \xrightarrow{b} q_b$ for some processes q_a and q_b. Since $p \underline{\leftrightarrow} q$, there must exist p_a and p_b such that $p \xrightarrow{a} p_a$ and $p \xrightarrow{b} p_b$, but this contradicts $|\mathcal{A}(p)| = 1$.

If $q_1 \not\underline{\leftrightarrow}_* q_2$, then $q_1 \not\underline{\leftrightarrow}_> q_2$ for some priority order $>$. Since we already know that $|\mathcal{A}(q)| = 1$, $q \to q_1$ and $q \to q_2$ implies $q \to_> q_1$ and $q \to_> q_2$. Hence there exist processes p_1 and p_2 such that $p \to_> p_1$ and $p \to_> p_2$ with $p_1 \underline{\leftrightarrow}_> q_1$ and $p_2 \underline{\leftrightarrow}_> q_2$. However, since p is uniformly 1-determinate, we know that $p_1 \underline{\leftrightarrow}_> p_2$, so $q_1 \underline{\leftrightarrow}_> q_2$, which is a contradiction.

If $\mathrm{norm}\,(q_1) \neq 1$, then we know from $p \underline{\leftrightarrow} q$ and $q \to q_1$ that $p \to p_1$ for some process p_1 with $p_1 \underline{\leftrightarrow} q_1$. But this implies $\mathrm{norm}\,(q_1) = \mathrm{norm}\,(p_1) = 1$, which is a contradiction. The argument for $\mathrm{norm}\,(q_2) \neq 1$ is similar.

Inductive step: Assume that q is uniformly k'-determinate for all $k' < k$. We now prove that q is also uniformly k-determinate. Assume towards a contradiction that q is not k-determinate. Then there must exist some q' such that $q \to^k q'$ and either

$|\mathcal{A}(q')| > 1$ or there are q_1 and q_2 such that $q' \to q_1$ and $q' \to q_2$, but either $q_1 \not\underline{\leftrightarrow}_* q_2$, $\mathrm{norm}\,(q_1) \neq 1$, or $\mathrm{norm}\,(q_2) \neq 1$.

The cases of $|\mathcal{A}(q')| > 1$, $\mathrm{norm}\,(q_1) \neq 1$, and $\mathrm{norm}\,(q_2) \neq 1$ are essentially the same as for the base case, except that one first gets a process p' such that $p \to^k p'$, and then reasons as before on p'.

We now consider the case of $q_1 \not\underline{\leftrightarrow}_* q_2$. This implies that $q_1 \not\underline{\leftrightarrow}_> q_2$ for some priority order $>$. Since $p \underline{\leftrightarrow}_* q$, we also get $p \underline{\leftrightarrow}_> q$, and since we know that q is uniformly k'-determinate for every $k' < k$. $q \to^k q'$ implies $q \to_>^k q'$. Therefore there exists a process p' such that $p \to_>^k p'$ and $p' \underline{\leftrightarrow}_> q'$. Since we already know that $|\mathcal{A}(q')| = 1$, $q' \to q_1$ and $q' \to q_2$ implies $q' \to_> q_1$ and $q' \to_> q_2$. Hence there exist p_1 and p_2 such that $p' \to_> p_1$ and $p' \to_> p_2$ as well as $p_1 \underline{\leftrightarrow}_> q_1$ and $p_2 \underline{\leftrightarrow}_> q_2$. However, since p is uniformly k-determinate, we know that $p_1 \underline{\leftrightarrow}_> p_2$, so we get $q_1 \underline{\leftrightarrow}_> q_2$, which contradicts our assumption.

□

C.2 Proof of Proposition 3

Proof of Proposition 3. Since our notion of uniformly k-determinate implies that of k-determinate in [3], Lemma 18 of that paper gives the result. □

D Proofs of results in Section 5

D.1 Proof of Proposition 4

Proof of Proposition 4. Notice that it is enough to prove that $A_n(\Theta(a + b)) \approx P_n$, since then

$$A_n(\Theta(a + b)) \approx P_n \implies P_n + A_n(\Theta(a + b)) \approx P_n + P_n \approx P_n.$$

Let $>$ be an arbitrary preorder. We now proceed by a case analysis on the behaviour of $A_n(\Theta(a + b))$ with respect to $>$.

- If $a > b$, then $A_n(\Theta(a + b)) \underline{\leftrightarrow}_> A_n(a)$.
- If $b > a$, then $A_n(\Theta(a + b)) \underline{\leftrightarrow}_> A_n(b)$.
- If a and b are incomparable, then $A_n(\Theta(a + b)) \underline{\leftrightarrow}_> A_n(a + b)$.

In any case, we conclude that $A_n(\Theta(a + b)) \underline{\leftrightarrow}_> P_n$. □

D.2 Proof of Lemma 6

Before proceeding to the proof, we recall a preliminary result on $\underline{\leftrightarrow}_>$. For a given priority order $>$, the bisimulation equivalence $\underline{\leftrightarrow}_>$ behaves like a classic bisimulation and therefore the following Lemma holds. (The same result on BCCSP processes was provided as Proposition 9 in [3]).

Lemma 8. *Consider processes p, q, assume $p \underline{\leftrightarrow}_> q$, for some priority order $>$ over $\mathcal{A}$, and let $k \in \mathbb{N}$. Then:*

1. *For every process p' s.t. $p \to_>^k p'$, there is a process q' s.t. $q \to_>^k q'$ and $p' \underline{\leftrightarrow}_> q'$.*
2. *$\mathcal{A}_>^k(p) = \mathcal{A}_>^k(q)$ so, in particular, $\mathcal{A}_>^1(p) = \mathcal{A}_>^1(q)$.*

Proof of Lemma 6. We first prove that $\mathcal{A}^k(p) = \{a\}$ for $0 \leq k < n$. Assume $p \sqsubseteq_* P_n$, which means that $p + r \underline{\leftrightarrow}_* P_n$ for some r, which in particular implies that $p + r \underline{\leftrightarrow} P_n$. By Lemma 8, we infer $\mathcal{A}^k(p + r) = \mathcal{A}^k(P_n) = \{a\}$. Since, moreover, $\mathcal{A}^k(p) \subseteq \mathcal{A}^k(p + r)$, we get $\mathcal{A}^k(p) = \{a\}$.

We proceed by contradiction. Let $1 \leq k < n$ be the least number such that p is not uniformly k-determinate. Then there exist processes p', p_1, and p_2 such that $p \to^k p'$, $p' \to p_1$, and $p' \to p_2$, and $p_1 \not\underline{\leftrightarrow}_* p_2$, or $\mathrm{norm}\,(p_1) \neq 1$, or $\mathrm{norm}\,(p_2) \neq 1$.

If $\mathrm{norm}\,(p_1) \neq 1$, then $p \to^k p'$ and $p' \to p_1$, so there exists P'_n and P''_n such that $P_n \to^k P'_n$ and $P'_n \to P''_n$ with $p_1 \underline{\leftrightarrow} P''_n$. But then $\mathrm{norm}\,(p_1) = \mathrm{norm}\,(P''_n) = 1$, which is a contradiction. A similar argument holds when $\mathrm{norm}\,(p_2) \neq 1$.

If $p_1 \not\underline{\leftrightarrow}_* p_2$, then $p_1 \not\underline{\leftrightarrow}_> p_2$ for some specific priority order $>$. Notice that since $|\mathcal{A}^i(p)| = \{a\}$ for all $0 \leq i < n$, we get that $p \to^k p'$, $p' \to p_1$, and $p' \to p_2$ implies $p \to_>^k p'$, $p' \to_> p_1$, and $p' \to_> p_2$. Since $p+r \underline{\leftrightarrow}_> P_n$ for some r, there exist P'_n, P''_n, and P'''_n such that $P_n \to_>^k P'_n$, $P'_n \to_> P''_n$, and $P'_n \to_> P'''_n$ with $p_1 \underline{\leftrightarrow}_> P''_n$ and $p_2 \underline{\leftrightarrow}_> P''_n$. Since $\mathrm{norm}\,(p_1) = 1 = \mathrm{norm}\,(p_2)$, we also get $\mathrm{norm}\,(P''_n) = 1 = \mathrm{norm}\,(P'''_n)$. However, we see from the definition of P_n that P'_n has a unique successor with norm 1. Hence it follows that $P''_n = P'''_n$, so $p_1 \underline{\leftrightarrow}_> P''_n = P'''_n \underline{\leftrightarrow}_> p_2$, which contradicts $p_1 \not\underline{\leftrightarrow}_> p_2$. $\square$

D.3 Proof of Proposition 5

Proof of Proposition 5. We start by noticing that since $\sigma(t)$ is uniformly Θ-n-dependent, by Definition 4 there are processes $p_0, \ldots, p_n$ s.t. $\sigma(t) = p_0 \to \ldots \to p_n$, $\mathrm{norm}\,(p_i) = 1$ for all $i = 0, \ldots, n-1$, and p_n is Θ-dependent. Since, moreover, we have $\mathrm{depth}\,(t) < n$, by Proposition 2 there are a process term t' and a string w s.t. $t \xrightarrow{s_1 \ldots s_h}_w t'$ and there are a variable x, an $l \in \mathbb{N}$ and a process q s.t. $t' \xrightarrow{x_s} \odot^l(x_d)$, $\sigma(x) \to^{n-k} q$, and $\odot^l(q) = p_n$.

Notice that, by Lemma 4, $t' \xrightarrow{x_s} \odot^l(x_d)$ is the same as $x \lhd_l t'$. Since, moreover, p_n is Θ-dependent, it must be the case that $|\mathcal{A}| > 1$. We can then apply Theorem 1, thus obtaining that there are a process term u' and an $m \in \mathbb{N}$ s.t. $u \to^k u'$ and $x \lhd_m u'$. Using again Lemma 4, $x \lhd_m u'$ is the same as $u' \xrightarrow{x_s} \odot^m(x_d)$. Notice that $\sigma(u) \sqsubseteq_* P_n$ implies that $\mathcal{A}^{n-1}(\sigma(u)) = \{a\}$. Hence, we have that a is locally maximal with respect to any priority order. Thus, from $\sigma(x) \xrightarrow{a}^{n-k} q$ and $u' \xrightarrow{x_s} \odot^m(x_d)$, Lemma 3 implies $\sigma(u') \xrightarrow{a}^{n-k} \odot^m(q)$. Hence we can infer that there are processes $q_0, \ldots, q_n$ s.t. $\sigma(u) = q_0 \to \ldots \to q_n = \odot^m(q)$. As p_n is Θ-dependent, $l > 0$ and thus, by Theorem 1, we can infer that $m > 0$, so that also $\odot^m(q)$ is Θ-dependent.

To conclude, we need to show that $\mathrm{norm}\,(q_i) = 1$ for each $i = 0, \ldots, n-1$. First of all we notice that, since $\sigma(t) \underline{\leftrightarrow}_* \sigma(u)$ and $\mathrm{norm}\,(\sigma(t)) = 1$, then $\mathrm{norm}\,(\sigma(u)) = \mathrm{norm}\,(q_0) = 1$. Moreover, since by the hypothesis $\sigma(t)$ is uniformly k-determinate for all $1 \leq k < n$, by Lemma 5 we infer that also $\sigma(u)$ is uniformly k-determinate for the same values of k, and thus $\mathrm{norm}\,(q_i) = 1$ for all $i = 1, \ldots, n-1$ is guaranteed by Definition 4. We can therefore conclude that $\sigma(u)$ is uniformly Θ-n-dependent. $\square$

$$(e_1)\,\frac{}{t \approx t} \qquad (e_2)\,\frac{t \approx u}{u \approx t} \qquad (e_3)\,\frac{t \approx u \quad u \approx v}{t \approx v} \qquad (e_4)\,\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$$

$$(e_5)\,\frac{t_1 \approx u_1 \quad t_2 \approx u_2}{t_1 \cdot t_2 \approx u_1 \cdot u_2} \qquad (e_6)\,\frac{t_1 \approx u_1 \quad t_2 \approx u_2}{t_1 + t_2 \approx u_1 + u_2} \qquad (e_7)\,\frac{t \approx u}{\Theta(t) \approx \Theta(u)}$$

Table 4: Rules of equational logic over BPA_Θ.

D.4 Proof of Theorem 2

Before proceeding to the proof of our main result, we report, in Table 4, the rules of equaitonal logic over BPA_Θ. As in operational semantics, they allow us to infer equations by proceeding inductively over the structure of terms. Let E be a sound set of axioms. Rules (e_1)-(e_4) are common for all process languages and they ensure that E is closed with respect to reflexivity, symmetry, transitivity and substitution, respectively. Rules (e_5)-(e_7) are tailored for BPA_Θ and they ensure the closure of E under BPA_Θ contexts. They are therefore referred to as the *congruence rules*. Briefly, rule (e_5) is the rule for sequential composition and it states that whenever $E \vdash t_1 \approx u_1$ and $E \vdash t_2 \approx u_2$, then we can infer $E \vdash t_1 \cdot u_1 \approx t_2 \cdot u_2$. Rule (e_6) deals with the nondeterministic choice operator in a similar way and rule (e_7) ensures that the priority operator preserves the equivalence of terms.

As elsewhere in the literature, we assume without loss of generality that for each axiom in E also the symmetric counterpart is in E, so that the symmetry rule is not necessary in the proofs, and that substitutions rules are always applied first in equational proofs, which means that the substitution rule $\frac{t \approx u}{\sigma(t) \approx \sigma(u)}$ may only be used over axioms $t \approx u$ in E. If this is the case, then $\sigma(t) \approx \sigma(u)$ is called a *substitution instance* of the axiom.

Moreover, we will make use of the following technical result from [3].

Lemma 9 ([3, Lemma 14]). *If $p \underline{\leftrightarrow}_* q$ and p is Θ-dependent, then so is q.*

We are now ready to prove our main result.

Proof of Theorem 2. As briefly discussed in Section 2, without loss of generality, we can disregard the symmetry rule in our inductive proof below by assuming that $u \approx t \in E$ whenever $t \approx u \in E$. Furthermore, we can assume that all applications of the substitution rule in derivations have a process equation from E as premise. This means that we only need to consider a new rule stating that all substitution instances of process equations in E are derivable, rather than considering the axiom rule — which states that all process equations in E are derivable —, and the substitution rule — which states that if a process equation is derivable, then so are all its substitution instances — separately.

We will now present the inductive argument over the number of steps in a proof of an equation $p \approx q$ from E. We proceed by a case analysis on the last rule applied to obtain $E \vdash p \approx q$.

Case 1*: reflexivity and transitivity.* In these cases, the proof follows immediately or by the induction hypothesis in a straightforward manner.

Case 2*: variable substitution.* Assume that $E \vdash p \approx q$ is the result of a closed substitution instance of an open process equation $t \approx u \in E$, namely there exists a substitution σ such that $\sigma(t) = p$ and $\sigma(u) = q$. Since $t \approx u \in E$, we have that $\mathrm{depth}(t), \mathrm{depth}(u) < n$. Moreover, from $p, q \sqsubseteq_* P_n$ it follows that $\mathcal{A}^*(p) = \mathcal{A}^*(q) = \{a\}$ and that, by Lemma 6, p and q are uniformly k-determinate for all $k \in \{1, \ldots n-1\}$. Hence by Proposition 5, we can conclude that if p is uniformly Θ-n-dependent, then so is q.

Case 3*: congruence rule.* We can distinguish three cases:

- The last rule applied in $E \vdash p \approx q$ is the congruence rule for the nondeterministic choice $+$. Then there exist closed process terms p_1, p_2, q_1 and q_2 such that $p = p_1 + p_2$, $q = q_1 + q_2$, $E \vdash p_1 \approx q_1$ and $E \vdash p_2 \approx q_2$ by shorter proofs. Since p is uniformly Θ-n-dependent, there must exist a process p' such that $p \to^n p'$, where p' is Θ-dependent and every process along the transitions from p to p' has norm 1. We can distinguish four possible subcases, regarding how such property is derived:
  1. p_1 is Θ-n-dependent.
  2. p_2 is Θ-n-dependent.
  3. $\mathrm{norm}(p_2) = 1$, $\mathrm{norm}(p_1) \neq 1$, and there are processes $p_1^1, \ldots, p_1^n$ such that $p_1 \to p_1^1 \to p_1^n = p'$ and p_1^n is Θ-dependent.
  4. $\mathrm{norm}(p_1) = 1$, $\mathrm{norm}(p_2) \neq 1$, and there are processes $p_2^1, \ldots, p_2^n$ such that $p_2 \to p_2^1 \to p_2^n = p'$ and p_2^n is Θ-dependent.

  In cases (1) and (2) we can immediately apply the induction hypothesis obtaining, respectively, that either q_1 or q_2 is Θ-n-dependent, and thus that q is Θ-n-dependent as well.

  The cases (3) and (4) require more attention. We detail only the proof for case (3), since the one for case (4) is symmetric. Firstly, we notice that since $p, q \sqsubseteq_* P_n$ then by Lemma 6 both p and q are uniformly k-determinate for all $k \in \{1, \ldots, n-1\}$. This implies that p_1 is uniformly k-determinate for the same values of k. Moreover, as $E \vdash p_1 \approx q_1$ gives $p_1 \underline{\leftrightarrow}_* q_1$ and $\mathrm{depth}(p_1) = n$, by Lemma 5 we obtain that also q_1 is uniformly k-determinate for $k \in \{1, \ldots, n-1\}$. Then, by Proposition 3 we can infer that there is a a process q_1^n such that $q_1 \to^n q_1^n$ and $q_1^n \underline{\leftrightarrow}_* p_1^n$, which, by Lemma 9, implies that q_1^n is Θ-dependent. Furthermore, uniform k-determinacy ensures that all the processes $q_1^1, \ldots, q_1^{n-1}$ in the sequence $q_1 \to q_1^1 \to \ldots \to q_1^{n-1} \to q_1^n$ have norm 1. Finally, we notice that since $\mathrm{norm}(p_2) = 1$ and $E \vdash p_2 \approx q_2$ implies $p_2 \underline{\leftrightarrow}_* q_2$, we can infer that $\mathrm{norm}(q_2) = 1$. By combining the properties of q_1 and q_2, we can conclude that $q = q_1 + q_2$ is uniformly Θ-n-dependent.
- The last rule applied in $E \vdash p \approx q$ is the congruence rule for the sequential composition. This means that $p = p_1 \cdot p_2$, $q = q_1 \cdot q_2$, $E \vdash p_1 \approx q_1$ and $E \vdash p_2 \approx q_2$ by shorter proofs. This case is vacuous, as $\mathrm{norm}(p) \geq 2$ and therefore p cannot be uniformly Θ-n-dependent.

– The last rule applied in $E \vdash p \approx q$ is the congruence rule for the priority operator Θ. Then there exist p' and q' such that $p = \Theta(p')$, $q = \Theta(q')$, and $E \vdash p' \approx q'$ by a shorter proof. Since p is uniformly Θ-n-dependent, there exists a sequence of processes $p = \Theta(p') \to \Theta(p_1) \to \cdots \to \Theta(p_{n-1}) \to \Theta(p_n)$ such that $\mathrm{norm}\,(\Theta(p_1)) = \ldots \mathrm{norm}\,(\Theta(p_{n-1})) = 1$ and $\Theta(p_n)$ is Θ-dependent. Note that, since $\Theta(p_n)$ is Θ-dependent, $|\mathcal{A}(p_n)\,| \geq 2$. Moreover, from the operational rules for Θ, $p' \to p_1 \to \cdots \to p_{n-1} \to p_n$ and from the definition of norm, $\mathrm{norm}\,(p_1) = \cdots = \mathrm{norm}\,(p_n) = 1$. From $E \vdash p' \approx q'$, we derive that $p' \underline{\leftrightarrow}_* q'$. Hence, $p' \underline{\leftrightarrow} q'$ holds and therefore we get a sequence $q' \to q_1 \to \cdots \to q_n$ such that $p_n \underline{\leftrightarrow} q_n$, which implies that $|\mathcal{A}(q_n)\,| \geq 2$. Thus, we infer $q = \Theta(q') \to \Theta(q_1) \to \cdots \to \Theta(q_n)$ and, since $|\mathcal{A}(q_n)\,| \geq 2$, $\Theta(q_n)$ is Θ-dependent. It remains to show that $\mathrm{norm}\,(\Theta(q')) = \mathrm{norm}\,(\Theta(q_i)) = 1$ for each $i \in \{1, \ldots, n-1\}$. As $q \sqsubseteq_* P_n$, by Lemma 6 we gather that q is uniformly k-determinate for all $1 \leq k < n$, from which it follows that $\mathrm{norm}\,(\Theta(q_i)) = 1$ for all $i \in \{1, \ldots, n-1\}$. Since, moreover, $p \underline{\leftrightarrow}_* q$ and $\mathrm{norm}\,(p) = 1$, we get $\mathrm{norm}\,(q) = 1$ and we conclude that q is Θ-n-dependent.

□